MILLAR

Use of the Compass
in
Geometrical Construction

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USE OF THE COMPASS IN GEOMETRICAL CONSTRUCTION

BY

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THESIS

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Mr. A. V. Millar B.S., 1897.

ENTITLED Use of the Compass in Geometrical Construction.

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Master of Science.

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HEAD OF DEPARTMENT OF Mathematics

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P R E F A C E.

The Italian mathematician, Lorenzo Mascheroni, in his work, "La geometria del compasso," (Pavia 1797) has proven that all geometrical constructions can be made by means of the compass alone. This work was translated into the French by Carrette and from the French into the German by Grueson under the title "Gebrauch des Zirkles" (Berlin 1825). This problem has been very ably treated by Frishauf in his work, "Die geometrischen Constructionen von L. Mascheroni und J. Steiner", (Groz 1869) and also by Hutt in "Die Mascheroni'schen Konstrucktionen" (Brandenburg 1880).

The corresponding problem, i.e., to make all geometrical constructions by means of the straight line alone, was first partly solved by the French mathematician, Brianchon in his "Application de la theorie des transversales" (Paris 1818). J. Steiner in his article "Die geometrischen Constructionen, ausgefuehrt mittelst der geraden Linie und eines festen Kreises" (Berlin 1833) has completely solved this problem with the help of a fixed circle.

These results are not generally known, and the object of the present work is to take the former problem, solve it and give a large number of examples as illustration of this method of construction. The work naturally divides itself into three sections as follows:

I. To determine what constructions are possible with the straight line and the compass together.

II. To prove that all these constructions can be made by means of the compass alone.

III. Examples. As examples for illustration, we take the construction problems in Wentworth's "Plane and Solid Geometry", revised edition.

With this as an outline we shall proceed with the development of it.

All references are to Wentworth's "Plane and Solid Geometry, Revised 1899".

Straight lines are determined by two points.

CHAPTER I.
GENERAL DISCUSSION OF THE PROBLEM.

SECTION I.

To determine what constructions are possible with the straight line and compass.

1. In order to determine what constructions are possible with the straight edge and compass, we shall consider the possibilities of the case analytically. Take the two equations

$$a_1 x + b_1 y + c_1 = 0$$

$$ax^2 + ay^2 + \cancel{bxy} + cx + dy + e = 0,$$

which are the general equations of the straight line and the circle, and therefore, for particular values of the constants, represent analytically any single operation which can be made with the straight edge and the compass respectively. The mathematical magnitudes which can be constructed by operating with these equations would, at most, include only those involving operations of addition, subtraction, multiplication and division and the extraction of the square root. This latter enters only when the roots of the second equation are involved. When these roots are imaginary they are to be excluded, for imaginary quantities are not represented by ordinary geometrical constructions. We may conclude, therefore, that all the operations which we can perform by means of the straight edge and the compass are=

I. All rational operations.

II. The extracting of the square root of any number.

If now it can be shown that a l l the rational operations, as well as the extraction of the square root of any given quantity, can be performed by the compass alone, we have demonstrated our proposition.

SECTION II.

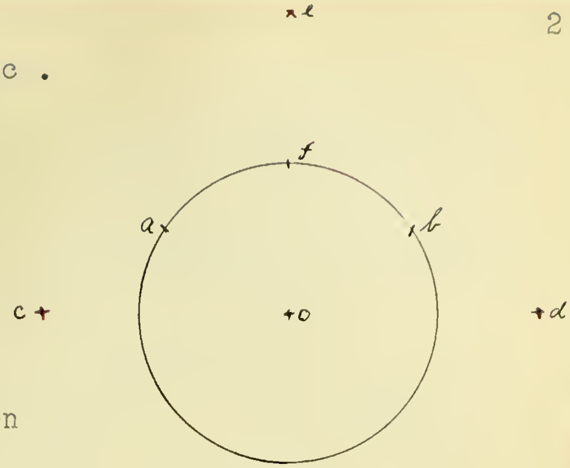
To prove that all of the above mentioned constructions can be made with the compass alone.

2. In order to construct all of the rational operations mentioned in the last section, it will be necessary from time to time to make auxiliary constructions, such as bisecting a given arc, etc. These constructions will be introduced as they are needed.

3. T o b i s e c t a g i v e n a r c .

Let ab be the given arc of a circle whose centre is o.

To bisect ab.



With o and b as centres and ob and ab respectively as radii, describe arcs intersecting at d. Similarly with o and a as centres and oa and ab as radii locate the point c. Then aboc and abdo are parallelograms. (W., Art., 182) Since od is parallel to ab is parallel to oc, then cd is a straight line, (W., Art., 105) and o is its middle point. Const.

With c and d as centres and radius cb, describe arcs intersecting at e. Then eo is perpendicular to cd at its middle point. (W., Art., 161)

Hence oe is perpendicular to ab, (W., Art., 107)

and must bisect the arc ab at some point f. (W., Art., 245)

To determine the point f, we have from the parallelogram aboc

$\overline{cb}^2 + \overline{ao}^2 = 2\overline{ab}^2 + 2\overline{ob}^2$, (W., Ex., 280)

or $\overline{cb}^2 = 2\overline{ab}^2 + \overline{ob}^2$.

But $\overline{cb}^2 = \overline{ce}^2 = 2\overline{ab}^2 + \overline{ob}^2$, Const.

and $\overline{co}^2 = \overline{ab}^2$.

Then $\overline{ce}^2 - \overline{co}^2 = \overline{oe}^2 = \overline{ab}^2 + \overline{ob}^2$,

also $\overline{cf}^2 = \overline{oc}^2 + \overline{of}^2 = \overline{ob}^2 + \overline{ab}^2$.

Hence $\overline{oe}^2 = \overline{cf}^2$ or oe=cf.

Therefore to determine the point f, make

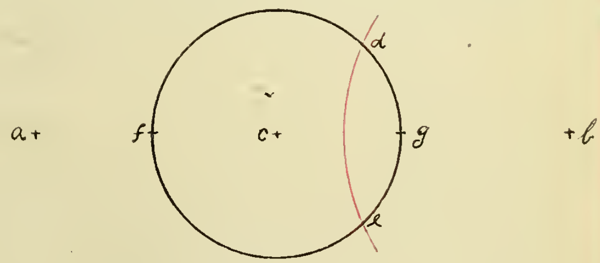
cf=df=oe.

4. T o f i n d t h e i n t e r s e c t i o n o f a s t r a i g h t l i n e a n d c i r c l e .

I. W h e n t h e l i n e p a s s e s t h r o u g h t h e c e n t e r o f t h e c i r c l e .

Let ab be the given line and c the given circle.

To find the intersection of ab with the circle c.



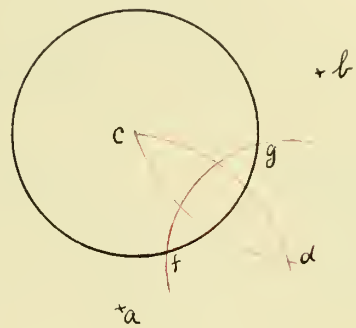
With b as a centre and any convenient radius strike an arc de. Then acb is perpendicular to de at its middle point, (W., Art., 264)

and therefore bisects the arcs de and dfe. (W., Art., 245)

By means of Article 3,bisect the arcs dge and dfe at the points g and f. Then g and f are the required points.

*Cor. By means of this article we can lay off any distance on a given straight line.

II. When the line does not pass through the centre of the circle.

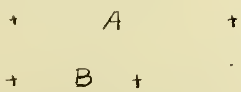


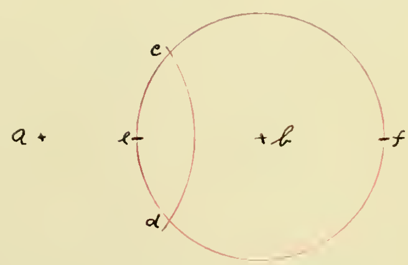
Let ab be the given line and c the given circle.

To find the intersections of ab with the circle c.

With a and b as centres and ac and bc respectively as radii describe arcs intersecting at d. With d as a centre and the same radius as the given circle,describe an arc cutting the given circumference at f and g. The two lines ab and fg are both perpendicular to cd at its middle point, (W.,Art.,161) and must therefore be the same line. Therefore f and g are the required points.

5. Given two magnitudes represented by the straight lines A and B; required to construct A+B and A-B.





Let A and B be the given lines.

To construct A+B and A-B.

Take the line ab equal to A and with b as a centre and a radius equal to B,describe a circle.

Find the intersections e and f of this circle with the line ab (Art.,4) Then

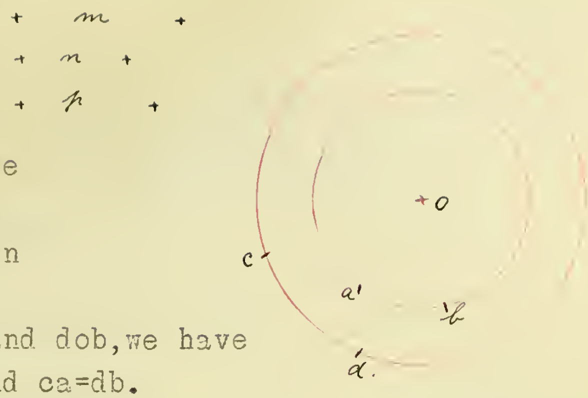
$$ae=ab-be=A-B,$$
$$af=ab+bf=A+B.$$

Therefore ae and af are the required lines.

6. To construct a fourth proportional to three given straight lines.

Let m,n and p be the three given straight lines.

To construct a fourth proportional to m,n and p.



With m and n as radii and o as a centre, describe two concentric circles. Lay off cd=p. With c as a centre and any convenient radius cut the inner circumference at a and with d as a centre and the same radius cut it at b. Then ab is the required line.

Proof= In the triangles coa and dob, we have

co=od, oa=ob and ca=db.

Therefore, the triangles coa and dob are similar, (W., Art., 358)
and hence $\angle coa=\angle dob$. (W., Art., 351)
Adding $\angle aod$ to both sides of this equation, we get $\angle cod=\angle aob$. Hence triangles aob and cod are similar, (W., Art., 357)
and we have

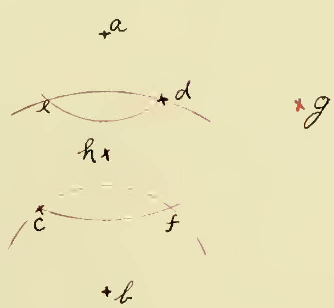
oc:oa=cd:ab

or m:n=p:ab.

Therefore ab is the required line.

*Cor. If p is greater than 2m, take equal multiples of m and n for radii.

7. Find the point of intersection of two straight lines.



Let ab and cd be the given straight lines.

To find the point of intersection of ab and cd

With a as a centre and ad and ac as radii, and b as a centre and radii bc and bd, describe arcs intersecting at e and f. With d and f as centres and radii respectively equal to cf and dc, describe arcs intersecting at g. By means of the last article, construct a fourth proportional to eg, ed, and fg. With e and d as centres and this fourth proportional as a radius, describe arcs intersecting at h. Then h is the required point.

Proof: ab is perpendicular to ed at its middle point, (W., Art., 161)
and since eh=dh, then h lies on ab. (W., Art., 160)

Since	eg:ed=fg:hd	Const.
Then	fg is parallel to hd.	(W., Art., 345)
But	fg is parallel to cd;	Const.

hence h must lie on cd. Therefore h is the intersection of ab and cd

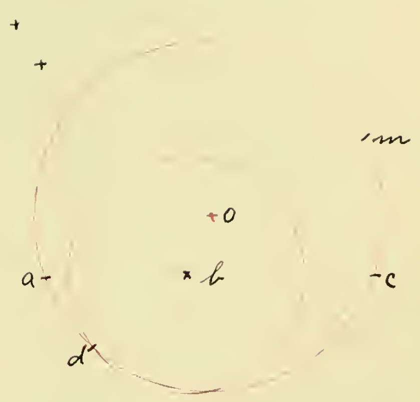
and is the required point.

8. Given two magnitudes represented by the straight lines A and B; find their product.

Let A and B be the given lines.

To construct A×B.

Let ab=A and bc=B and construct ac=A+B. (Art.,5)



With b as a centre and unit radius describe a circle. Let d be any point in the circumference of this circle not in the line ac. Pass a circle through the three points a,d and c(W.,Art.,258) Find the intersection,m,of the line bd with this circle.

Them bm is the required product.

Proof: bm×bd=ab×bc. (W.,Art.,378)

But bd=1 Const.

Then bm=ab×bc=A×B.

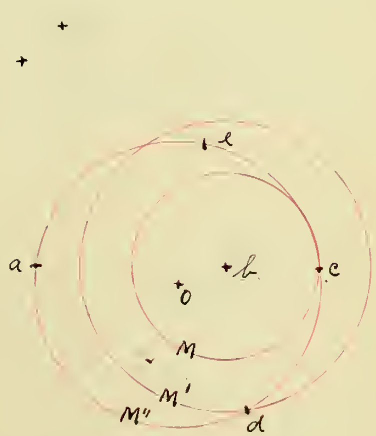
Therefore bm is the required line.

*Cor. If A=B,then bm=A^2. By continuing the operation we can raise a number to any required power.

9. Given two magnitudes represented by the straight lines A and B; find their quotient.

Let A and B be the given lines.

To construct A/B



Let ab=A. With b as a centre and a radius equal to unity,describe a circle M',and let d be any point in the circumference of M' not in the line ab.

Pass a circle,M",through the three points,a,d and c. (W.,Art.,258)

Find the intersection,e,of bd with this circle. (Art.,4,II.) Then b is the required line.

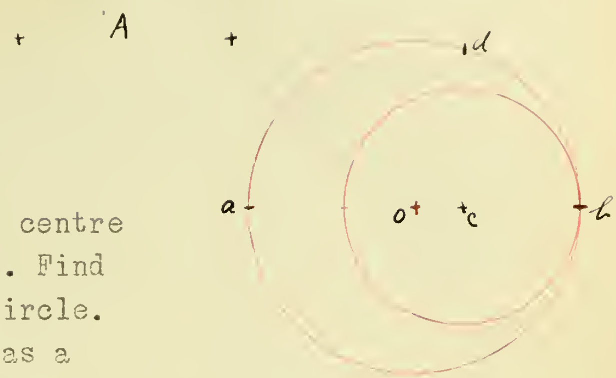
Proof: ab×bc=bd×be. (W.,Art.,378)

But bc=1 .

Then ab=bd×be or A=B×be and be=A/B .

Therefore be is the required line.

10. Given a magnitude represented by the straight line A; construct the square root of A.



Let A be the given line.

To construct the square root of A.

Let the given line, A, be laid off in the position ac. With c as a centre and a unit radius, construct a circle. Find the intersection, b, of ac with this circle. (Art.4, I) Construct a circle on ab as a diameter. (W., Art., 302) Erect a perpendicular to the line ab at the point c, intersecting this circumference at d. (W., Art., 301) Then cd is the required line.

Proof:	ac:cd=cd:cb.	(W., Art., 370)
But	cb=1.	Const.
Then	$\overline{cd}^2$=ac=A or cd=$\sqrt{A}$.	

Therefore cd is the required line.

11. We have now constructed by means of the compass alone all the rational operations (addition, subtraction, multiplication and division) and the extraction of the square root. In section I we found that these are all operations which can be constructed by means of both the compass and the straight edge. We have therefore demonstrated our proposition that all the constructions which can be made by means of both the compass and the straight edge can be made by means of the compass alone.

CHAPTER II.

APPLICATIONS.

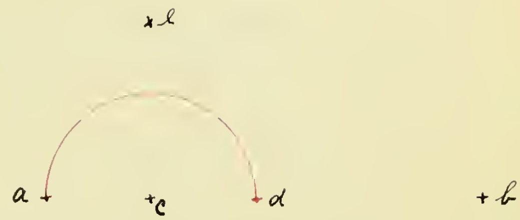
The examples of the present chapter are the construction problems of Wentworth's "Plane and Solid Geometry", Revised 1899.

12. *To erect a perpendicular to a line at a given point.*

I. *When the point is on the line.*

Let ab be the given line and c the given point.

To erect a perpendicular to ab at c.



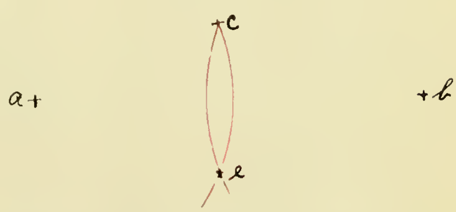
With c as a centre and ca as a radius, describe a circle and find its intersection, d, with ab. (Art., 4, I)
With a and d as centres and any convenient radius, describe arcs intersecting at e. Then ec is the required perpendicular.

Proof:	ac=cd,	(W., Art., 217)
and	ae=de.	Const.
Therefore ec is perpendicular to ab.		(W., Art., 161)

II. *When the point is without the line.*

Let ab be the given line and c the given point.

To erect a perpendicular to ab from c.

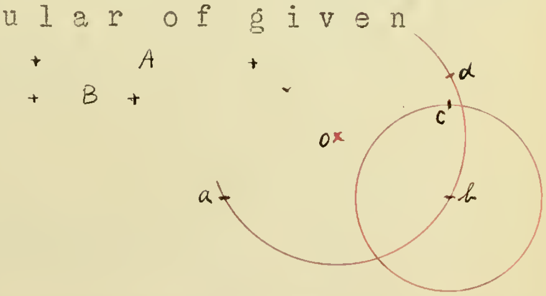


With a and b as centres and ac and bc respectively as radii, describe arcs intersecting at e. Then ec is the required perpendicular. (W., Art., 161)

13. *Given a line A, to construct at its end point a perpendicular of given length B.*

Let A and B be the given lines.

To construct a perpendicular of length B at the end point of A.



Take the line ab equal to A.

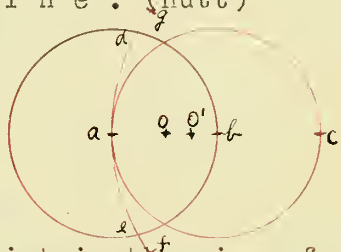
With a and b as centres and any convenient radius, describe arcs intersecting at o. With o as a cent-

re and the same radius, describe a circle. Find the intersection, d, of this circle with ao (Art., 4, I). With b as a centre and B as a radius, describe a circle and find its intersection, c, with bd. (Art., 4, I). Then bc is the required perpendicular.

Proof: bd is perpendicular to ab (W., Art., 290)
and bc=B. Const.
Therefore bc is the required perpendicular.

*Cor. 1. From the right triangle abc, we get $\overline{ac}^2=\overline{ab}^2+\overline{bc}^2$, i.e. $\overline{ac}^2=A^2+B^2$. By extracting the square root of both sides of the equation, we have $ac=\sqrt{A^2+B^2}$. By taking A as the diameter of a circle and from one extremity of it laying off B as a chord, we would get $ac=\sqrt{A^2-B^2}$.

14. T o b i s e c t a g i v e n l i n e . (Hutt)



+ A +

Let A be the given line.
To bisect A.
With a as a centre and A as a radius, describe a circle and let b be any point in the circumference. With c as a centre and co as a radius, strike an arc cutting the first circumference at d and e. With e and d as centres and radius A, describe arcs intersecting at o. Then ab, and therefore A, is bisected at o.

Proof: Triangles ado and adc are similar. (W., Art., 357)
Hence $ao:ad=ad:ac$,
and therefore $ao=\overline{ad}^2/ac=\overline{ab}^2/2ab=ab/2=A/2$.
Therefore A is bisected at o.

*Cor. T o b i s e c t o b .
Make ag=af=bd and go'=fo'=bd.
Then ob is bisected at o'.

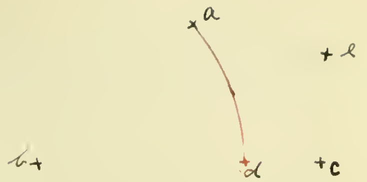
Proof: Triangles ago' and agc are similar. (W., Art., 357)
Then $ao':ag=ag:ac$ therefore $ao'=\overline{ag}^2/ac=\overline{bd}^2/ac$.
Since ac is bisected at b, we have
$$\overline{ad}^2+\overline{cd}^2=2\overline{ab}^2+2\overline{bd}^2. \quad \text{(W., Art., 377)}$$
But $\overline{ad}^2+\overline{cd}^2=5\overline{ab}^2$.
Hence $5\overline{ab}^2=2\overline{ab}^2+2\overline{bd}^2$ or $\overline{bd}^2=3/2\ \overline{ab}^2$.
substituting we have
$$ao'=(3/2\ \overline{ab}^2)/2ab=3/4\ ab.$$
Therefore $bo'=1/4\ ab$.
Proceeding in a similar manner we can bisect bo'.

15. To bisect a given angle.

Let abc be the given angle.

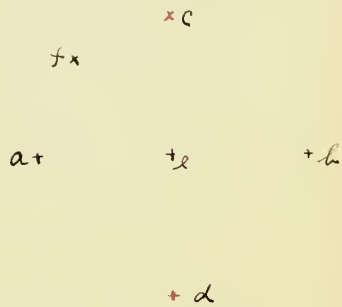
To bisect the angle abc.

With b as a centre and ba as a radius,describe an arc and find where it interesects bc. (Art.,4,I). With a and d as centres and any convenient radiusdescribe arcs intersecting at e. Then be bisects the given angle abc. (W.,Art.,245)



16. To construct an angle of 45° and 135°.

Take any straight line,ab,and erect a perpendicular,cd,at its middle point. (Art.,12,I)
Find the intersection,e,of ab and cd. (Art.,7)
Bisect the angle aec by the line ef. (Art.,15)
Then $\angle$aef=1/2 $\angle$aec=1/2 of 90°=45°,
and $\angle$bef=180°-$\angle$aef=135°
Therefore aef and bef are the required angles.



17. Construct an equilateral triangle, having given one side.

Let a be the given side.

To construct an equilateral triangle,having its sides equal to a.

Take any line,bc,equal to a.
With b and c as centres and a as a radius,strike arcs intersecting at d.
Then bcd is the required triangle for each side is equal to a.
*Cor. $\angle$dbc=1/3 of 180°=60° (W.,Art.,146)
Draw be perpendicular to bd at b. (Art.,12,I)
Then $\angle$cbe=60°+90°=150°

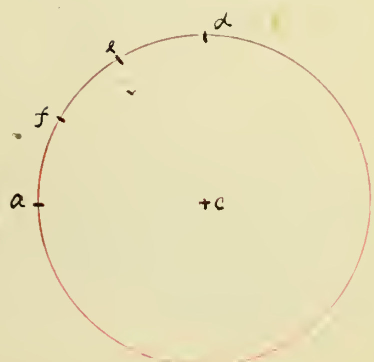
18. To trisect a right angle.

Let acd be the given right angle.

To trisect the angle acd

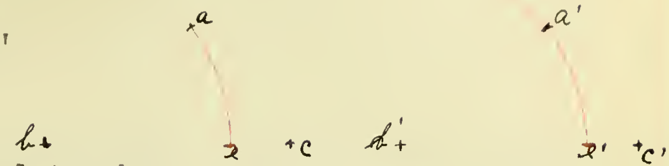
With a and d as centres and ac as a radius cut the circumference at e and f.
Then the angle acd is trisected by e and f.

Proof:	$\angle$acd=90°	Hypothesis.
and	$\angle$ace=60°.	(Cor.,Art.,17)
Therefore	$\angle$ecd=90°-60°=30°.	
Similarly	$\angle$acf=30°,	
and therefore	$\angle$ecf=30°.	



Hence ∠acd is trisected.

19. At a given point in a given straight line to construct an angle equal to a given angle.



Let abc be the given angle, b'c' the given line and b' the point on the line.

To construct at b' an angle equal to abc.

With b as a centre and ba as a radius, strike an arc and find its intersection, e, with bc. (Art., 4, I). With b' as a centre and the same radius strike an arc cutting b'c' at e'. With e' as a centre and ae as a radius strike an arc locating the point a'. Then a'b'c' is the required angle.

Proof: Chord ae=chord a'e'. Const.

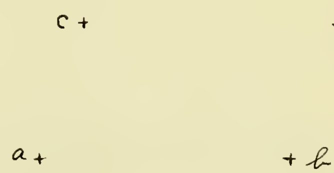
Then ∠a'b'c'=∠abc. (W., Arts., 243, 237)

Therefore a'b'c' is the required angle.

20. To draw a straight line parallel to a given straight line through a given external point.

Let ab be the given line and c the given point.

To draw a line through c parallel to ab.



With b and c as centres and ac and ab respectively as radii, describe arcs intersecting at d. Then cd is the required line.

Proof: ab=cd and ac=bd, Const.

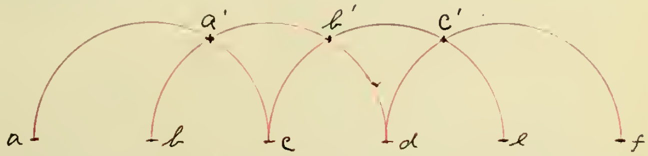
therefore abdc is a parallelogram. (W., Art., 182)

Hence cd is parallel to ab and is the required line.

21. To multiply a given line by any positive integer.

Let ab be the given line.

To multiply ab by any positive integer.



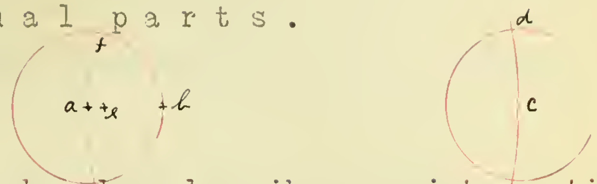
With b as a centre and ba as a radius, draw a circle and find its intersection, c, with ab. (Art., 4, I). Now with c as a centre and the same radius, draw a circle cutting ac at d. This can be continued indefinitely.

Then ac=2ab. (W., Art., 217)

Similarly ad=3ab etc.

*Cor. The points a'b'c' etc.,form a similar series of points,with a'b'=ab.

22. *To divide a given straight line into a given number of equal parts.*



Let ab be the given straight line.

To divide ab into equal parts.

Construct ac=n×ab. (Art.,21)

With a and c as centres and radii ab and ac,describe arcs intersecting in d and f. With f and c as centres and radii ab and fd respectively,describe arcs intersecting at e. Then ae=ab/n.

Proof:Both ac and ec are parallel to fd. (W.,Art.,182)

Then e must lie on the line ac.

The triangles afe and afc are similar (W.,Art.,357)

Therefore ae:af=af:ac

or ae:ab=ab:ac=1:n

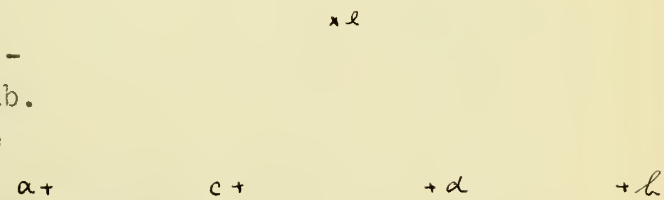
Hence ae=ab/n

23. *To construct an equilateral triangle having given the perimeter.*

Let ab be the given perimeter.

To construct an equilateral triangle,having given the perimeter,ab.

Divide ab at c and d into three equal parts.(Art.,22).



Construct an equilateral triangle,having cd as a side.(Art.,17)

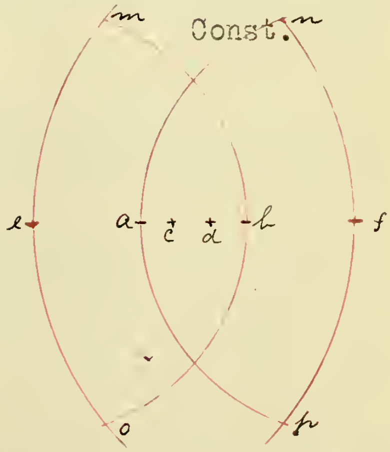
Then cde is the required triangle.

Proof: Triangle cde is equilateral. Const.

Its perimeter is 3×cd=ab.

Therefore cde is the required triangle.

Second method.



Let ab be the given perimeter.

To construct an equilateral triangle, having ab as a perimeter.

Make ae=bf=ab (Art.,21)

With e and f as centres and radii eb and ef,describe arcs intersecting at m.n.o and p. With o and m as centres and radius eb,describe arcs intersecting in c,and with the same radius and n and p as centres,describe arcs intersecting in d. Then ab is trisected at c and d,and an equilateral triangle with ac as a side will be the required triangle.

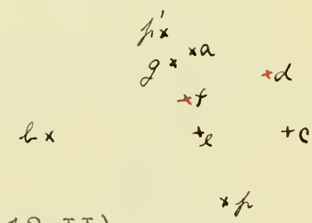
Proof: The points a,c,d,b lie in the same straight line (W.,Art.,160)
The triangles emc and emf are similar. (W.,Art.,357)
Then em:ec=ef:em=3:2
Hence em=2ab=3/2(ab+ac) or ab=3ac
Similarly ab=3bd or ab=3cd
Therefore the triangle constructed with ac as a side is the required triangle.

24. To divide a line into four equal parts by two different methods.

A line may be divided into four equal parts by applying article 14 twice. In that article, if we had bisected ao as we did bo, then ab would have been divided into four equal parts.

This problem may be solved by using article 22. In this case n=4 and ac=4ab

25. Through a given point to draw a line which shall make equal angles with the two sides of a given angle.



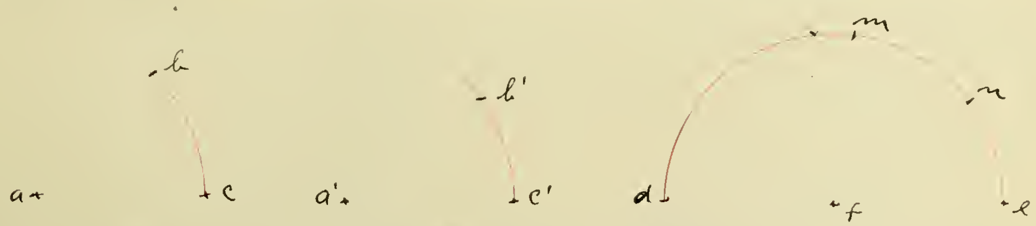
Let abc be the given angle and p the given point.

To draw a line through p making equal angles with ab and cb.

Draw the bisector, bd, of the angle abc. Construct pp' perpendicular to bd. (Arts.,15 & 12,II) Then pp' is the required line.

Proof: The right triangles bgf and bef are equal. (W.,Art.,142)
Hence ∠bgf=∠bef. (W.,Art.,128)
Therefore pp' is the required line.
*Cor. This also solves the problem: To draw a line through a given point so that it shall form with the sides of a given angle an isosceles triangle.

26. To find the third angle of a triangle when two of the angles are given.



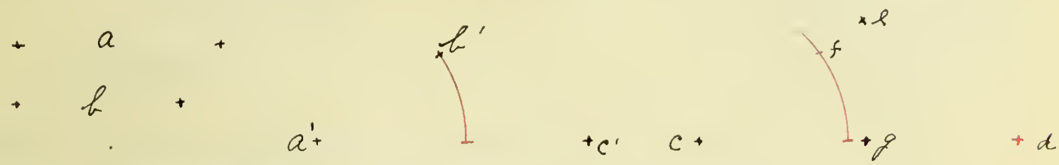
Let bac and b'a'c' be the two given angles.
To construct the third angle of the triangle.

At any point, f, in the line de construct

$\angle efn = \angle b'a'c'$ and $\angle nfm = \angle bac$. (Art., 19)

Then $\angle dfm$ is the required angle. (W., Art., 139)

27. To construct a triangle when two sides and the included angle are given.



Let a and b be the given sides and b'a'c' the given angle.

To construct a trianlge having the given sides and angles.

On the line cd, construct $\angle fcg = \angle b'a'c'$. (Art., 19)

With c as a centre and radius a, strike an arc and find its intersection e, with cf. (Art., 4, I)

Similarly lay off cg equal to b.

Then ceg is the required triangle.

Proof: $\angle fcg = \angle b'a'c'$, ce=a and cg=b. Const.

Therefore cge is the required angle. (W., Art., 143)

28. To construct a triangle when a side and two angles of the triangle are given.

Let de be the given side and abc and a'b'c' the given angles.

To construct a triangle having de for one of its sides and abc and a'b'c' as two of its angles.

On any line mn lay off mo=de. (Art., 4, I)

At m and o respectively construct $\angle pmo = \angle abc$ and $\angle p'om = \angle a'b'c'$. (Art., 19)

Find the intersection, p", of mp and op'. (Art., 7)

Then mop" is the required angle.

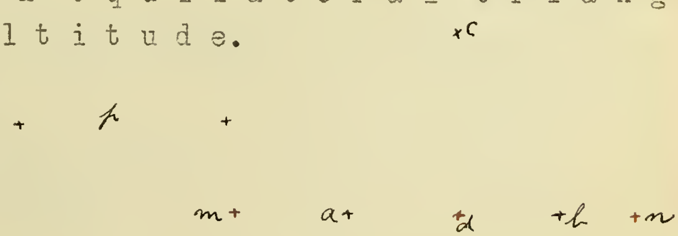
Proof: mo=de, $\angle p''mo = \angle abr$ and $\angle p''om = \angle a'b'c'$. Const.

Therefore p"om is the required triangle. (W., Art., 139)

29. To construct an equilateral triangle having given the altitude.

Let p be the given altitude.

To construct an equilateral triangle having p for its altitude.



At any point d of the line mn, draw dc perpendicular to mn and lay it off equal to p. (Arts.,12,4,I)
At c, construct $\angle acd=\angle bcd=30°$ (Art.,18)
and find the intersections, a and b, with mn. (Art.,7)
Then abc is the required triangle.

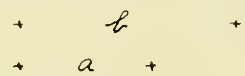
Proof: The altitude dc=p. Const.
Tri.adc=tri.bdc. (W.,Art.,142)
Hence $\angle dac=\angle dbc$. (W.,Art.,127)
But $\angle acb=60°$ Const.
and therefore $\angle dac+\angle dbc=2\angle dac$ or $2\angle dbc=180°-60°=120°$,
or $\angle dac=\angle dbc=60°$.
then triangle abc is equiangular and therefore equilateral.(W.,Art.,148)
Therefore abc is the required angle.

30. To construct an isosceles triangle having given:

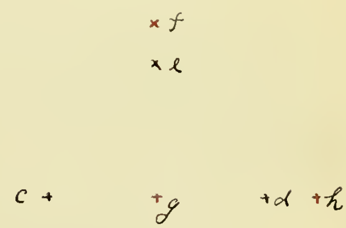
I. The base and altitude.

Let b be the given base and a the given altitude.

To construct an isosceles triangle with base b and altitude a.



On any line ch, lay off cd=b (Art.,4,I) and erect a perpendicular fg at its middle point, (Art.,12,I)
Lay off ge=a. (Art.,4,I)

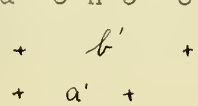


Then cde is the required triangle.

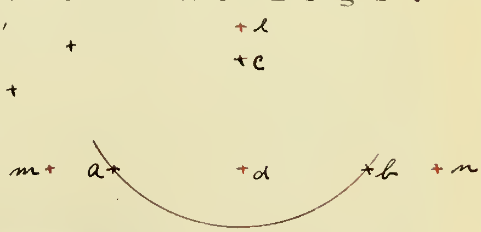
Proof: The altitude ge=a Const.
and ce=de. (W.,Art.,160)
Therefore cde is the required triangle.

II. The altitude and one of the legs.

Let b' be the given leg and a' the given altitude.



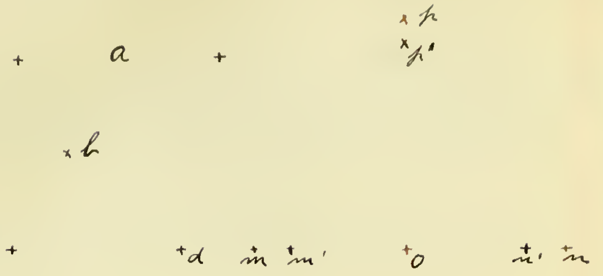
To construct an isosceles triangle having a' as an altitude and b' as one of the equal sides.



At any point in the line mn, erect a perpendicular de. (Art.,12,I)
Lay off de=a' (Art.,4,II)
With c as a centre and a radius b', strike an arc and find its points of intersection a and b, with mn. (Art.,4,II)
Then abc is the required triangle.

Proof: The altitude cd=a' and ac=bc=b'. Const.
Therefore abc is the required triangle.

III. The angle at the vertex and the altitude.



Let a be the given altitude and bcd the given angle at the vertex.

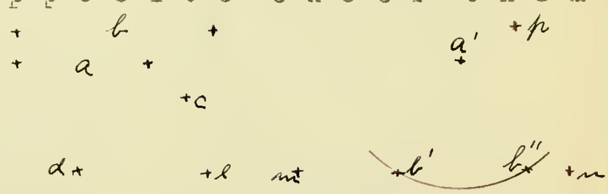
To construct an isosceles triangle whose altitude is a and whose vertical angle is bcd.

Erect op perpendicular to mn at any point and lay off op'=a. (Arts.,12,I;4,I)
Bisect the angle bcd, (Art.,15)
and at p' construct half the angle bcd on each side of op'.
Find the points of intersection, m' and n', of the sides of these angles with with the line mn. (Art.,7)
Then m'n'p' is the required triangle.

Proof:	The altitude op'=a	Const.
and	∠m'p'n'=∠bcd.	Const.
Also	Tri.m'op'=tri.n'op'	(W.,Art.,142)
and therefore m'p'=n'p'.		(W.,Art.,127)
Hence the triangle m'n'p' is isosceles.		(W.,Art.,120)

Therefore m'n'p' is the required triangle.

31. To construct a triangle when two sides and the angle opposite one of them are given.



Let a and b be the two given sides and cde the given angle opposite the side a.

To construct a triangle having a and b as two of its sides and the angle opposite a equal to cde.

At the point m in the line mn, construct angle pmn equal to angle cde. (Art.,19)
Lay off ma' equal to the given side b. (Art.,4,I)
With a as a centre and a as a radius, strike an arc and find its intersection, b' abd b", with mn. (Art.,4,II)
Then either ma'b' or ma'b" is the required triangle.

Proof: a=a'b'=a'b"; b=ma'; ∠cde=∠a'mb'. Const.

Therefore ma'b' or ma'b" is the required triangle..

*Cor. If a is of such a length that the arc is tangent to mn or if a=b we have but one triangle. If a is less than the distance from a' to mn, we have no construction.

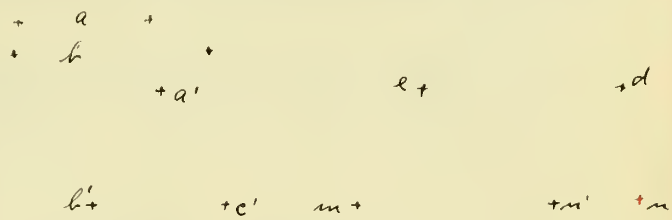
32. To construct a triangle when the three sides of the triangle are given.

The triangle may be constructed by laying any one of the given sides on any line, taking these two points as centres and the other two given sides as radii and strike arcs locating the third vertex of the required triangle.

33 To construct a parallelogram when two sides and the included angle are given.

Let a and b be the given sides and a'b'c' the given angle.

To construct a parallelogram having its two adjacent sides equal respectively to a and b and the included angle equal to a'b'c'.



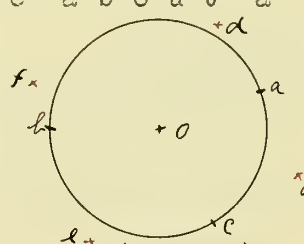
On any line, mn, lay off mn'=b and construct $\angle$emn'=$\angle$a'b'c'. (Arts., 4, Cor; 19) Lay off me=a. With e and n' as centres and radii equal respectively to b and a, describe arcs intersecting in d. Then mn'de is the required parallelogram.

Proof:	mn'=b, me=a and $\angle$emn'=$\angle$a'b'c',	Const.
also	ed=mn' and n'd=me,	Const.
hence	mn'de is the required parallelogram.	

34. To circumscribe a circle about a given triangle.

Let abc be the given triangle.

To circumscribe a circle about abc.



Erect perpendiculars ed and fg at the middle points of bc and ac. (Art.,12,I)
Find the intersection o of ed and fg. (Art.,7)
With o as a centre and oa as a radius, describe a circle. Then this is the required circle.

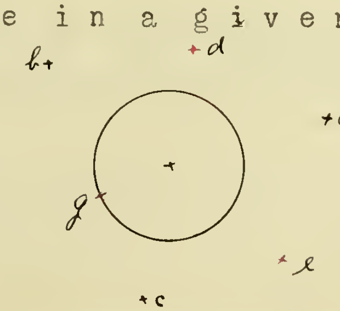
Proof: Since the point o lies on both lines ed and fg it is equally distant from a, b and c. (W., Art., 160)
Then the circle with o as centre and radius oa will pass through a, b and c. Therefore this is the required circle.

35. To inscribe a circle in a given triangle.

Let abc be the given triangle.

To inscribe a circle in abc.

Bisect the angles b and c by the

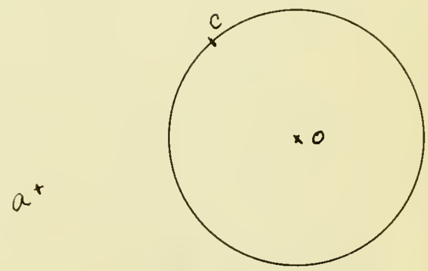


lines be and cd. (Art.,19)
Find the intersection o of cd and be. (Art.,7)
From o draw og perpendicular to bc. (Art.,12,II)
Find the intersection,g,of og and bc. (Art.,7)
With o as a centre and og as a radius,describe a circle.
Then this is the required circle.

Proof: Since o is on both be and cd it is equally distant from all three sides of the triangle abc. (W.,Art.,162)
Hence a circle described with o as a centre and og as a radius will touch all three sides of the triangle. Therefore this is the required circle..

36. Through a given point to draw a tangent to a given circle.

I. When the given point is on the circumference.

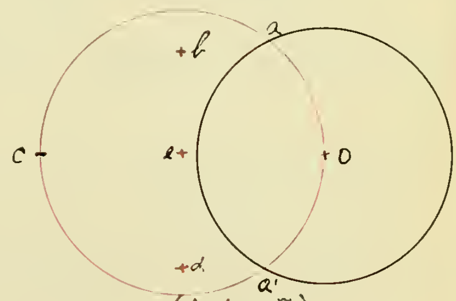


Let c be the point on the circumference whose centre is c.

To draw a tangent to the circumference at c.

Erect a perpendicular ac to oc at the point c. (Art.,12,I)
Then ac is the required tangent. (W.,Art.,253)

II. When the given point is without the circle.



Let o be the centre of the given circle and c the given point.

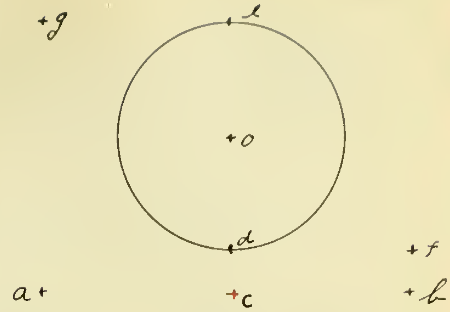
To draw a tangent to the circle whose centre is o from the given point c.

Erect a perpendicular,bd,at the middle point of oc. (Art.,12,I)
Find the intersection,e,of oc and bd. (Art.,7)
With e as a centre and ec as a radius,describe a circle cutting the given circumference at a and a'.
Then ac and a'c are the required tangents.

Proof: ca is perpendicular to oa and ca' to oa'. (W.,Art.,290)
Therefore ca and ca' are tangents to the given circle.(W.,Art.,253)

37. To draw a tangent to a given circle, so that it shall be parallel to a given straight line.

Let o be the centre of the given circle, and ab the given straight line.



To draw a tangent to the circle whose centre is o which is parallel to ab.

From c,draw oc perpendicular to ab.

Find the intersections,e and d,of oc with the circumference. (Art.,12,II) & (Art.,4,I)

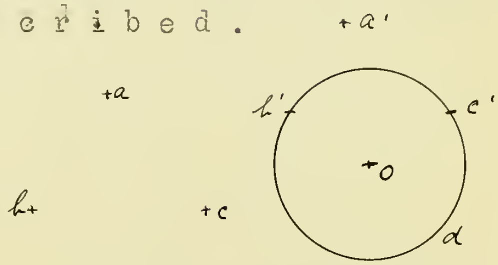
Draw tangents to the circle at the points e and d. (Art.,36,I)

Then these are the required tangents.

Proof:	oc is perpendicular to ab.	Const.
Also	ge and df are perpendicular to ce.	(W.,Art.,254)
Then	ge,df and ab are all parallel.	(W.,Art.,104)

Therefore ge and df are the required tangents.

38. U p o n a g i v e n s t r a i g h t l i n e , t o d e s c r i b e a s e g m e n t o f a c i r c l e i n w h i c h a g i v e n a n g l e c a n b e i n s c r i b e d .



Let abc be the given angle. and b'c' the given line.

To describe the segment of a circle on b'c' in which angle abc may be inscribed.

At b' construct ∠a'b'c'=∠abc (Art.,19)

Erect a perpendicular to a'b' and to b'c' at its middle point, (Art.,12,I) and find the intersection,o,of these perpendiculars. (Art.,7)

With o as a centre and ob' as a radius,describe a circle. Then b'dc' is the required segment.

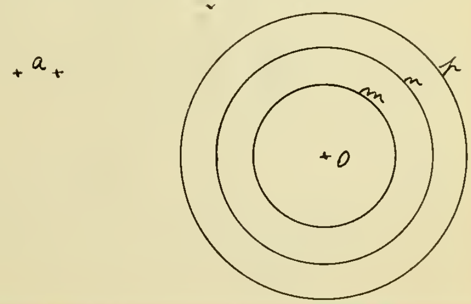
Proof: Any angle inscribed in the segment b'dc' is measured by one-half the acr b'c'. (W.,Art.,289)

Also ∠a'b'c'=∠abc is meusured by half the ard b'c'. (W.,Art.,295)

Then any angle inscribed in the segment b'dc' is equal to the angle abc.

Therefore b'dc' is the required segment..

39. F i n d t h e l o c u s o f a p o i n t a t a g i v e n d i s t a n c e f r o m a g i v e n c i r c u m f e r e n c e .



Let a be the given distance and n the given circumference with centre o.

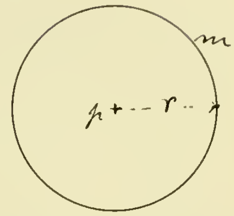
To find the locus of points at a distance a from the circumference n.

Let r be the radius of the cir-

cumference n and construct r+a and r-a. (Art.,5)
With o as a centre and r-a and r+a as radii, describe the circumference m and p. Then m and p are the required loci.

40. Find the locus of the centre of a circle,

I. Which has a given radius r and passes through a given point p.



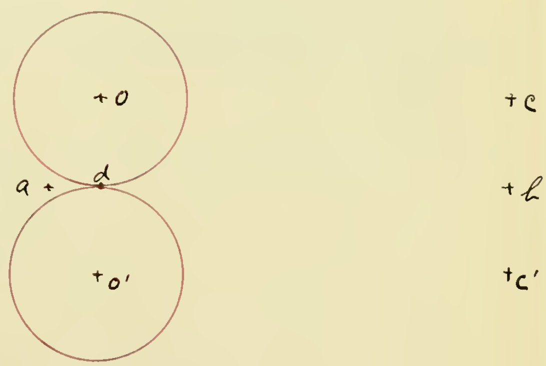
Let p be the given point and r the given radius.

To find the locus of the centre of the circle whose radius is r and whose circumference passes through p.

With p as a centre and r as a radius, describe a circumference m. Then this is the required locus.

Proof: If we take any point in m as a centre and r for a radius and describe a circle, the circumference of this circle will pass through p. Therefore m is the required locus.

II. Which has a given radius and touches a given line ab.



Let ab be the given line and r the given radius.

To find the locus of the centre of a circle whose radius is r and whose circumference touches the line ab.

At d, draw oo' perpendicular to ab. (Art.,12,I)

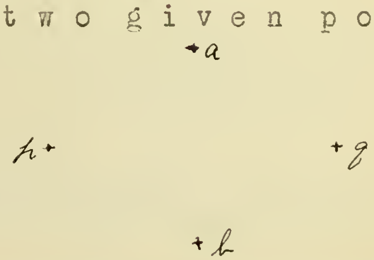
Lay off od=o'd=r. (Art.,4,Cor.)

Through o and c' draw oc and o'c' respectively parallel to ab. (Art.,20)
Then oc and o'c' are the required loci.

Proof: Every point in oc and o'c' is at a distance r from ab. (W.,Art.,181)

Then if we take any point in oc or o'c' as a centre and r as a radius and describe a circle, then the circumference of this circle will touch ab. Therefore oc and o'c' are the required loci.

III. Which passes through two given points p and q.



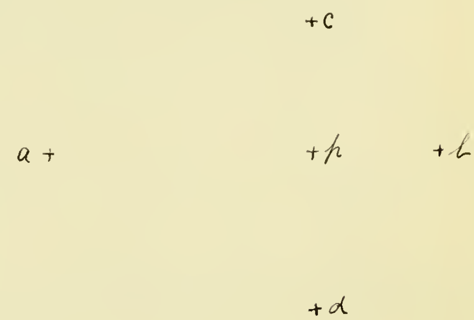
Let p and q be the two given points.

To find the locus of the centres of all circles whose circumferences pass through p and q.

Erect a perpendicular, ab, at the middle point of pq. (Art., 12, I)
Then ab is the required locus.

Proof: Any point in ab is equally distant from the two points p and q. (W., Art., 160)
Then any circumference which passes through the two points p and q will have its centre in the line ab.
Therefore ab is the required locus.

IV. Which touches a given straight line ab at a given point p.



Let ab be the given line and p the given point.

To find the locus of the centre of a circle which is tangent to ab at the point p.

Construct cd perpendicular to ab at p. (Art., 12, I)
Then cd is the required locus.

Proof: All circles whose centres are in cd and whose circumferences pass through p are tangent to ab at p. (W., Art., 253)
A circle whose centre is not on cd and whose circumference passes through p is not tangent to ab at p. (W., Art., 254)
Therefore cd is the required locus.

V. Which touches each of two given parallels.

Let ab and cd be the two given parallel lines.

To find the locus of the centre of a circle which touches ab and cd.

Construct ef midway between and parallel to ab and cd. (Art., 20)
Then ef is the required locus.

Proof: Since every point of ef is equally distant from ab and cd, a circle described with any point p of ef as a centre and the distance from p to ab or cd as a radius will touch both ab and cd. This will not be true for any centre outside of ef.
Therefore ef is the required locus.

VI. Which touches each of two intersecting lines.

Let ab and cd be the given intersecting lines.

To find the locus of the centre of a circle tangent to ab and cd.

Find the intersection, o, of ab and cd. (Art., 7)
Construct the bisectors ef and e'f' of the angles aoc and aod. (Art., 15)
Then ef and e'f' are the required loci.

Proof: The lines ef and e'f' are the loci of points equally distant from ab and cd. (W., Art., 162)
Then any circle tangent to ab and cd will have its centre in ef or e'f'.
Therefore ef and e'f' are the required loci.

41. To find in a given line a point x which is equidistant from two given points.

Let ab be the given line and c and d the given points.

To find a point on ab equidistant from c and d.

Construct ef perpendicular to cd at its middle point. (Art., 12, I)
Find the intersection x of ab and ef. Then x is the required point. (Art., 7)

Proof: Since x lies in ef it is equidistant from c and d. (W., Art., 160)
x also lies in ab and is therefore the required point.

42. To find a point x equidistant from three given points.

This problem is solved in article 35.

43. To find a point x equidistant from two given points and at a given distance from a third given point.

Let a and b be the first two given points, c the third and r the given distance.

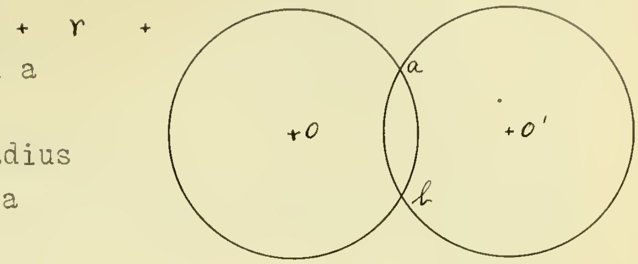
To find a point equidistant from a and b and at a distance r from c.

With c as a centre and r as a radius describe a circle. Construct ef perpendicular to ab at its middle point. (Art., 12, I)
Find its intersections, x and x', with the circle whose centre is c. (Art., 4, II)
Then x and x' are the required points.

Proof: Since the points x and x' are on the circumference of the circle, they are at the distance r from c. Since they lie in the line ef they are equidistant from a and b. (W., Art., 160)

Therefore x and x' are the required points.

44. To construct a circle which has a given radius and passes through two given points.



Let r be the given radius and a and b the given points.

To construct a circle with radius r passing through the given points a and b.

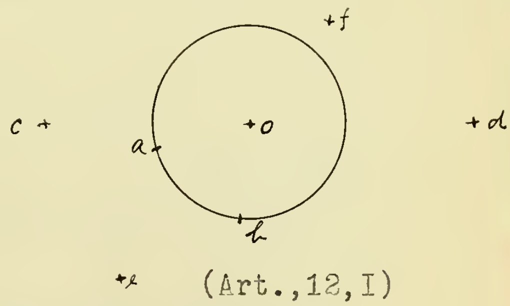
With a and b as centres and radius r, describe arcs intersecting at o and o'. With o and o' as centres and r as a radius, describe circles. Then these are the required circles.

Proof: Since bo=ao=r, then a circle described with o as a centre and r as a radius will pass through a and b. Similarly for o'. Therefore the circles whose centres are o and o' are the required circles.

45. To find a point x at a given distance from two given points.

The points o and o' of article 44 were so constructed.

46. To construct a circle which has its centre in a given line and passes through two given points.



Let cd be the given line and a and b the given points.

To construct a circle passing through a and b whose centre lies in cd.

Construct ef perpendicular to ab at its middle point. (Art., 12, I)

Find the intersection o of ef and cd (Art., 7)

With o as a centre and oa as a radius, describe a circle. Then this is the required circle.

Proof: Since o lies in ef it is equidistant from a and b. (W., Art., 160)

Then if we take o as a centre and oa as a radius the circle will pass through a and b. Therefore this is the required circle.

47. To find a point x equidistant from two given points and also equidistant from two intersecting lines.

Let p and p' be the given points and ab and cd the given intersecting lines.

To find a point x equidistant from the lines ab and cd and the points p and p'.

Find the intersection, o, of ab and cd. (Art., 7)
Construct the bisector, ef, of the angle bod. (Art., 15)
Construct e'f' perpendicular to pp' at its middle point. (Art., 12, I)
Find the intersection, x, of ef and e'f'. (Art., 7)
Then x is the required point.

Proof: Since x lies on ef it is equidistant from ab and cd. (W., Art., 162)
And since it lies on e'f' it is equidistant from p and p'. (W., Art., 160)
Therefore x is the required point.

48 To find a point x equidistant from two given points and also equidistant from two parallel lines.

Let p and p' be the given points and ab and cd the given parallel lines.

To find a point x equidistant from p and p' and also from ab and cd.

From a draw ac perpendicular to cd, (Art., 12, II)
and bisect ac at e. (Art., 14)
Through e, draw ef parallel to ab. (Art., 20)
Construct e'f' perpendicular to pp' at its middle point. (Art., 12, I)
Find the intersection, x, of ef and e'f'. (Art., 7)
Then x is the required point.

Proof: Since x lies on ef it is equidistant from ab and cd. (W., Art., 181)
and since it lies on e'f' it is equidistant from p and p'. (W., Art., 160)
Therefore x is the required point.

49. To find a point x equidistant from two given intersecting lines and also equidistant from two given parallels.

Let ab and a'b' be the two given parallels and cd and c'd' the two given intersecting lines.

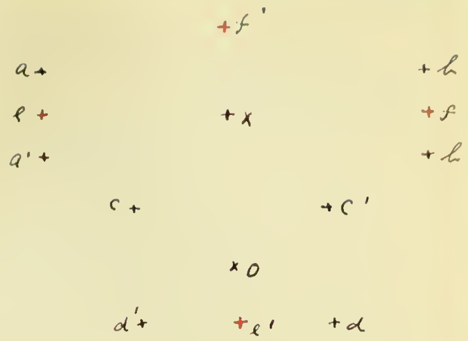
To find a point x equidistant from ab and a'b' and also equidistant

from cd and c'd'.

Find the intersection, o, of cd and c'd'. Construct the bisector, e'f', of the angle coc'. (Art., 7) & (Art., 15)

Draw ef parallel to ab and midway between ab and a'b' as in article 48. Find the intersection, x, of ef and e'f! (Art., 7)

Then x is the required point.

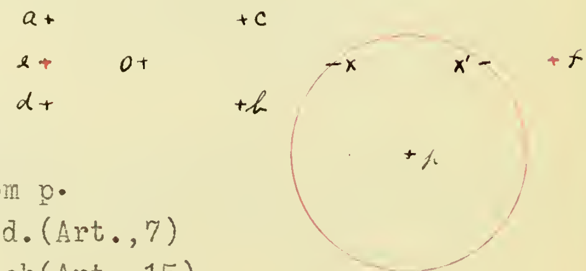


Proof: Since x lies on ef it is equidistant from ab and a'b' and since it lies on e'f' it is equidistant from cd and c'd' (W., Arts., 181, 162)

Therefore x is the required point.

50. To find a point x equidistant from two given intersecting lines and at a given distance from a given point.

Let r be the given distance, p the given point and ab and cd the given intersecting lines.



To find a point x equidistant from ab and cd and at the distance r from p.

Find the intersection, c, of ab and cd. (Art., 7)

Construct the bisector, ef, of the angle cob. (Art., 15)

With p as a centre and r as a radius, describe a circle and find its intersections, x and x', with ef. (Art., 4, 41)

Then x and x' are the required points.

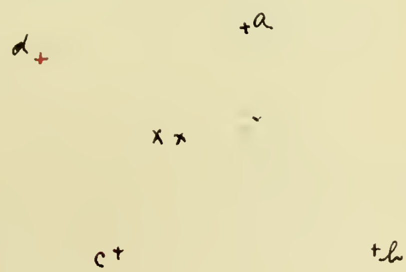
Proof: Since x and x' lie on the circumference of the circle, they are at a distance r from p and since they lie on ef they are equidistant from ab and cd. (W., Art., 162)

Therefore x and x' are the required points.

51. To find a point x which lies in one side of a given triangle and is equidistant from the other two sides.

Let abc be the given triangle.

To find a point x in ac equidistant from ab and bc.



Construct the bisector, bd, of the angle abc. (Art., 15)

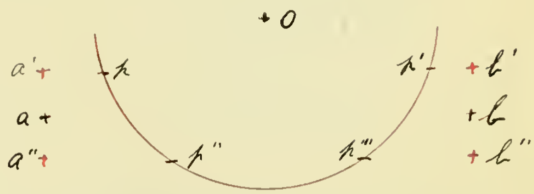
Find the intersection, x, of ac and bd. (Art., 7)

Then x is the required point.

Proof: Since x lies in bd it is equidistant from ab and bc. (W.,Art.,162)
It als c lies in ac. Therefore x is the required point.

52. A straight railway passes two miles from a town. A place is four miles from the town and one mile from the railway. To find by construction the places that answer this discription.

Let o be the town and ab the railway.



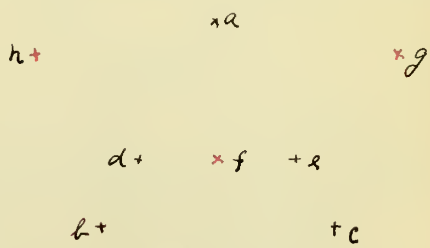
To find points four miles from o and one mile from ab.

Construct a'b' and a"b" parallel to ab and at a distance one from it as in article 48. With o as a centre and radius four, describe a circle. Find the intersections p,p',p", p'" of this circle with a'b' and a"b". (Art.,4,II)
Then p,p',p",p"' are the required points.

Proof: Since these points lie on the circumference of the circle, they are at the distance four from o, and since they lie on a'b' and a"b", they are at the distance one from ab. (W.,Art.,181)
Therefore these are the required points.

53..In a triangle abc, to draw de parallel to base bc, cutting the sides of the triangle in d and e, so that de shall equal db+ec.

Let abc be the given triangle.



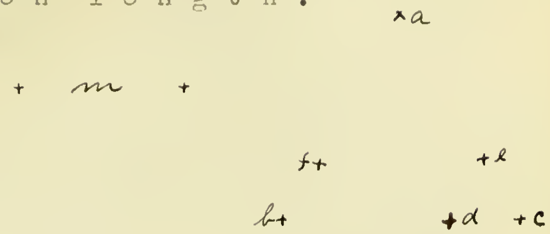
To draw de parallel to the base so that de=db+ec.

Construct the bisectors, ch and bg, of the angles acb and abc. (Art.,15)
Find the intersection, f, of ch and bg. (Art.,7)
Through f draw de parallel to bc. (Art.,20)
Find its intersections, d and e, with ab and ac. (Art.,7)
Then de is the required line..

Proof:	de is parallel to bc.	Const.
Then	$\angle$fbc=$\angle$bfd.	(W.,Art.,110)
But	$\angle$fbc=$\angle$fbd.	Const.
Hence	$\angle$fbd=$\angle$dfb.	
and therefore	df=db.	(W.,Art.,147)

Similarly fe=ec and therefore de=bd+ec. Therefore de is the required line.

54. To draw through two sides of a triangle a line parallel to the third side so that the part intercepted between the sides shall have a given length.



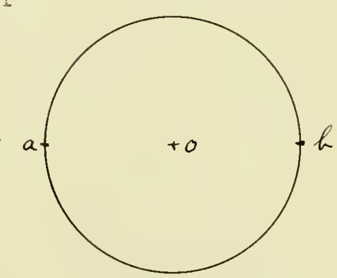
Let abc be the given triangle, and m the given length.

To draw a line parallel to bc so that the part, fe, intercepted between ab and ac shall have the given length m.

Lay off bd equal to m. (Art., 4, Cor.)
Through d, draw de parallel to ab. (Art., 20)
Find its intersection, e, with ac. (Art., 7)
Through e, draw ef parallel to bc. (Art., 20)
Then ef is the required line.

Proof: bdef is a parallelogram. Const.
Hence ef=bd=m. (W., Art., 178)
Therefore ef is the required line.

55. Prove that the locus of the vertex of a right triangle, having a given hypotenuse as base, is the circumference described upon the given hypotenuse as diameter.



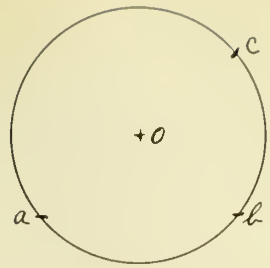
Let ab be the given hypotenuse.

To prove that the locus of the vertex of the right triangle, having ab as a hypotenuse is the circumference of a circle whose diameter is ab.

Proof: Bisect ab at o. (Art., 14)
Construct a circle with o as a centre and oa as a radius. Any triangle found with ab as one side and its vertex in the circumference is a right triangle. (W., Art., 290)
Therefore the locus of the vertex of a right triangle having ab as the hypotenuse is the circumference.

56. Prove that the locus of the vertex of a triangle, having a given base and a given angle at the vertex is the arc which forms with the base a segment capable of containing the given angle.

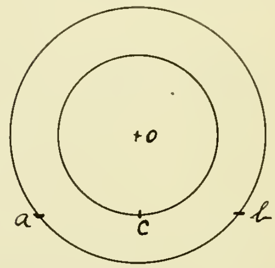
Let ab be the given base and acb the given angle.



To prove that the locus of the vertex of a triangle having ab as a base and acb as the angle at the vertex is the arc acb.

Proof: Construct a segment on ab in which the given angle at the vertex may be inscribed. (Art.,38)
Any triangle having ab as a base and its vertex in the arc of this segment will have the required angle at the vertex. Hence the arc of this segment is the locus of the vertex of a triangle having ab as a base and the given angle at its vertex.

57 Find the locus of the middle point of a chordof a given length that can be drawn in a given circle.

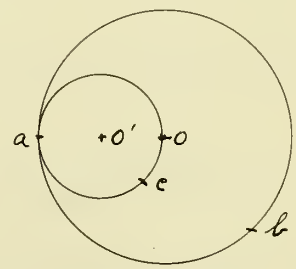


Let ab be the chord in the circle whose centre is o.

To find the locus of the middle point of ab.

Bisect ab at c. (Art.,14)
Then since equal chords are equally distant from the centre,the locus of the point c is the circumference of a circle whose centre is o and radius oc.

58. Find the locus of the middle point of a chord drawn from a given point in a given circumference.



Let c be the middle point of the chord ab in the circle whose centre is o.

To find the locus of c.

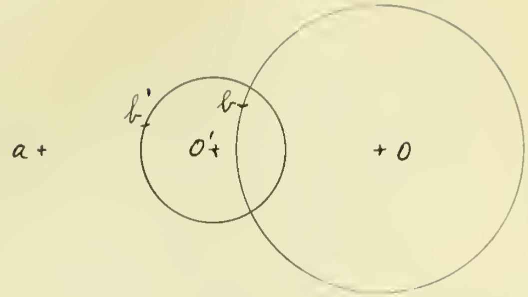
oc is perpendicular to ab.(W.,Art.,247)
Therefore the locus of the point c is the locus of the vertex of a right triangle having ao as a hypotenuse. By article 55 this is a circle having ao as a diameter. Bisect ao at o'. (Art.,14)
With o' as a centre and o'a as a radius,describe a circle. Then this is the required locus.

59. Find the locus of the middle point of a straight line drawn from a given exterior point to a given circumference.

Let a be the given external point,and o the centre of the given circumference.

To find the locus of the middle point of a line drawn froma to the given circumference.

Bisect ao at o'. (Art.,14)
With o' as a centre and a radius equal to one half of ob,describe a circle. Then this circumference whose centre is o' is the required locus.



Proof: Take any point,b,in the circumference whose centre is o and find the intersection b' of ab with the circumference whose centre is o'. (Art.,4,II)

Then ao:ao'::ob:o'b'::2:1. Const.
Then the triangles aob and ao'b' are similar. (W.,Art.,357)
Hence ao:ao'::ab:ab'::2:1.
Since b is any point on the given circumference and b' is the middle point of ab,then the circumference whose centre is o' is the required locus.

60. A straight line moves so that it remains parallel to a given line,and touches at one end a given circumference. Find the locus of the other end.

Let ab be the given fixed line, a'b' any position of the movable line and o the centre of the given circumference.

To find the locus of b' as a' moves on the given circumference and a'b' remains parallel to ab.

Construct the parallelogram oa'b'o'. (Art.,33)
With o' as a centre and oa' as a radius,describe a circle. Then this is the required locus.

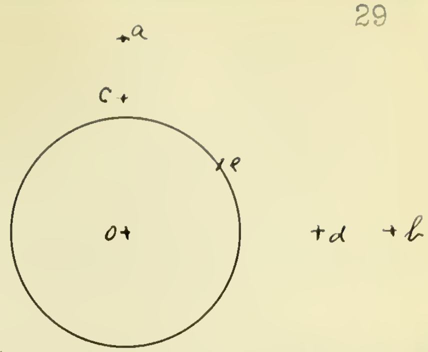
Proof: Since a'b' moves parallel to ab it will always remain parallel and equal to oo'. Hence oa'b'o' is always a parallelogram. (W.,Art.,183)
Hence o'b' is always equal to oa'. (W.,Art.,178)
Therefore the circumference whose centre is o' is the required locus.

61. A straight rod moves so that its ends constantly touch two fixed rods which are perpendicular to each other. Find the locus of its middle point.

Let ao and bo be the two given lines at right angles,cd the movable

line and e its middle point.

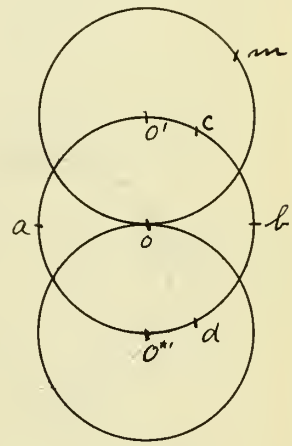
To find the locus of e as cd moves on ao and bo.



With o as a centre and ce as a radius describe a circle. The part of this circumference included between ao and bo is the required locus.

Proof: In every position of cd, we have oe=1/2 cd for oe is always the line joining the middle point of the hypotenuse of a right triangle to the vertex of the right angle which is half the hypotenuse. Therefore the part of the circumference between ao and bo is the required locus.

62. In a given circle let aob be a diameter, oc any radius, cd the perpendicular from c to ab. Upon oc take om equal to cd. Find the locus of the point m as oc turns about o.



Let ab be the given diameter, oc any radius, cd perpendicular to ab and om=cd.

To find the locus of the point m as oc turns about o.

Erect a perpendicular to ab at o (Art., 12, I) and find its intersections o' and o" with the circle. (Art., 4, I)
With o' and o" as centres and oa as a radius describe circles. Then these circumferences are the required locus.

Proof: In the two triangles oo'm and ocd, om=cd, oo'=od and ∠odc=∠o'om
Hence tri.oo'm=tri.ocd. (W., Art., 148)
Hence o'm=oc. Since ocm is any position of the radius, then the locus of m will be the circumference of a circle with o' as a centre and o'm as radius. When oc gets below ab the locus of m will be the circumference of a circle having o" as a centre and oa as a radius.

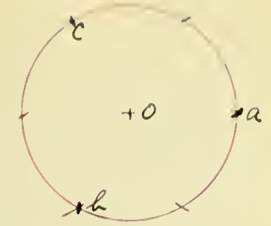
63. To construct an equilateral triangle, having given the radius of the circumscribed circle.

Let oa be the given radius.

To construct an equilateral triangle the radius of whose circumscribed circle is oa.

With o as a centre and oa as a radius describe a circle. Inscribe

a regular hexagon in this circle.(W.,Art.,469) Join the alternate vertices a,b,c of this hexagon and we have our required triangle.

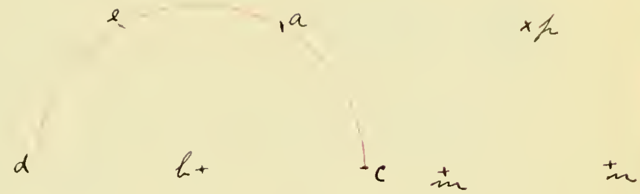


64. To construct an isosceles triangle, having given:

I. The angle at the vertex and the base.

Let mn be the given base and abc the given angle at the vertex.

To construct an isosceles triangle having mn for a base and abc for the angle at the vertex.



Extend the line cb to d. (Art.,4,Cor.)

Bisect the angle abd by be. (Art.,15)

At m and n construct $\angle pmn=\angle pnm=\angle abc$. (Art.,19)

Find the intersection p of mp and np. (Art.,7)

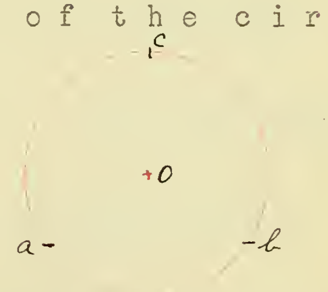
Then mnp is the required triangle.

Proof: Since $\angle m=\angle n$, the triangle mnp is isosceles. (W.,Art.,147)

$\angle p=180°-(\angle m+\angle n)=180°-2\angle abc=\angle abc$ Const.

Therefore mnp is the required triangle.

II. The base and the radius of the circumscribed circle.



Let ab be the given base and ao the radius of the circumscribed circle.

To construct an isosceles triangle having ab for a base and ao for the radius of the circumscribed circle.

With a and b as centres and ao as a radius, describe arcs intersecting at o. With o as a centre and oa as a radius, describe a circle. Erect a perpendicular to ab at its middle point. (Art.,12,I)

Find its intersection, c, with the circumference. (Art.,4,I)

Then abc is the required triangle.

Proof: ac=bc. (W.,Art.,160)

Hence triangle abc is isosceles. (W.,Art.,120)

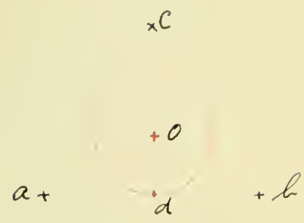
Therefore abc is the required triangle.

III. The base and the radius of the inscribed circle.

Let ab be the given base and od the radius of the inscribed circle.

To construct an isosceles triangle having ab as a base and od as the radius of the inscribed circle.

Erect a perpendicular at the middle point of ab. (Art.,12,I)
Lay off od equal to the given radius. (Art.,4,Cor.)
With o as a centre and od as a radius, describe a circle.



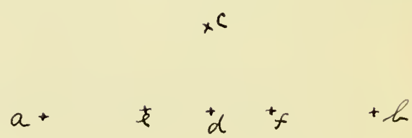
Draw tangents to this circle from a and b. (Art.,36,II)
Find their intersection c. (Art.,7)
Then abc is the required triangle.

Proof: tri.adc=tri.bdc. (W.,Art.,143)
Then ac=bc and tri.abc is isosceles. (W.,Art.,120)
Therefore abc is the required triangle.

IV. The perimeter and the altitude.

Let ab be the given perimeter and cd the given altitude.

To construct an isosceles triangle having ab as a perimeter and cd as an altitude.



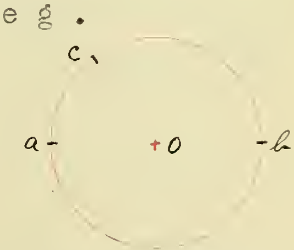
Erect a perpendicular at the middlepoint of ab. (Art.,12,I)
Lay off cd equal to the given altitude. (Art.,4,Cor.)
Erect perpendiculars at the middle points of ac and bc. (Art.,12,I)
Find their interesections, e and f, with ab. (Art.,7)
Then efc is the required triangle.

Proof: ae=ec and bf=cf. (W.,Art.,160)
Hence ec+cf+fe=ab.
Triangle ecf is also isosceles and is therefore the required triangle.

65. To construct a right triangle, having given:

I The hypotenuse and one leg.

Let ab be the hypotenuse and ac the given leg.



To construct a right triangle having ab for a hypotenuse and ac for one leg.

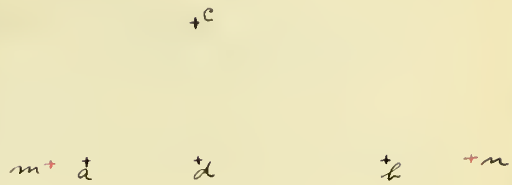
Construct a circle on ab as a diameter.
With a as a centre and the given leg as a radius, cut the circumference at c. Then abc is the required triangle.

Proof: acb is a right triangle. (W.,Art.,290)
ab is the given hypotenuse and ac is the given leg. Therefore abc is the required triangle.

II. One leg and the altitude upon the hypotenuse.

Let ac be the given leg and cd the given altitude.

To construct a right triangle with ac as one leg and cd as the altitude upon the hypotenuse.



At any point in the line mn construct a perpendicular dc. (Art.,12,I)
Lay it off equal to the given altitude. (Art.,4,Cor.)
With c as a centre and the given leg as a radius, strike an arc cutting mn at a. Erect a perpendicular to ac at c. (Art.,12,I)
Find its intersection, b, with mn. (Art.,7)
Then abc is the required triangle.

Proof: acb is a right angle. Const.
ac is the required leg and cd is the required altitude. Therefore abc is the required triangle.

III. The median and the altitude drawn from the vertex of the right angle.

Let oc be the median and od the altitude.

To construct a right triangle with oc as the median and od as the altitude drawn from the vertex of the right angle.

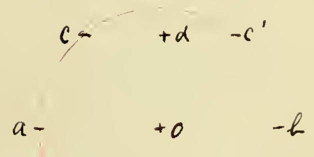
Construct a circle with the given median as a radius and to any diameter, ab, construct a perpendicular, od, at its middle point. (Art.,12,I)
Lay it off equal to the given altitude. (Art.,4,Cor.)
Draw a line through d parallel to ab (Art.,20)
and find its intersection, c and c', with the circle. (Art.,4,II)
Then abc and abc' are the required triangles.

Proof: od is the required altitude and oc the required median. Const.
Angles acb and ac'b are right angles. (W.,Art.,290)
Therefore acb and ac'b are the required triangles.

IV. The hypotenuse and the altitude upon the hypotenuse.

Let ab be the given hypotenuse and od the given altitude.

To construct a right triangle with ab for a hypotenuse and od for an altitude.



Construct a circle on ab as a diameter, erect a perpendicular at the middle point

of ab,and lay it off equal to the altitude od. (Arts.,12,I;4,Cor.)
Through d,draw cc' parallel to ab. (Art.,20)
Find its intersections c and c' with the circle. (Art.,4,II)
Then acb and ac'b are the required triangles.

Proof: They have the given hypotenuse and altitude. Const.
Angles acb and ac'b are right angles. (W.,Art.,290)
Therefore acb and ac'b are the required triangles.

V. The radius of the inscribed circle and one leg.

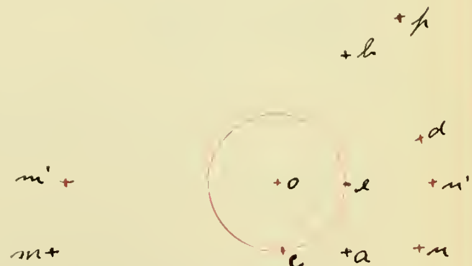
Let ab be the given leg and oe the given radius.

To construct a right triangle,having ab as one leg and oe as the radius of the inscribed circle.

Draw mn parallel to ab and at a distance oe from it.(Art.,20).
Construct ∠abd=45°. (Art.,16)
Find the intersection o of mn and bd. (Art.,7)
With o as a centre and oe as a radius,describe a circle.
Draw tangents to this circle from a and b and find their intersection c. (Art.,36,II)
Then abc is the required triangle.

Proof: abc is circumscribed about a circle of the given radius.Const.
Since ∠abd=45° then ∠abc=90°. (W.,Art.,261)
Therefore abc is the required triangle.

VI. The radius of the inscribed circle and an acute angle.



Let nmp be the give angle and oc the radius of the inscribed circle.

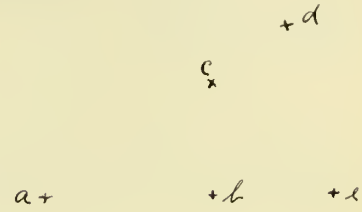
To construct a right triangle having nmp as one of the acute angles and oc as the radius of the inscribed circle.

Draw the bisector md of the angle nmp, (Art.,15)
Draw m'n' parallel to mn and at the distance oc from it. (Art.,20)
Find the intersection o of m'n' and md. (Art.,7)
With o as a centre and the given radius,describe a circle and find the point e where it cuts m'n'. (Art.,4,I)
Draw a tangent to the circle at e. (Art.,36,I)
Find its intersection a and b with mn and mp. (Art.,7)
Then mab is the required triangle.

Proof: By construction,mab is circumscribed about the given circle and also has the given acute angle.

ab is perpendicular to m'n' (W.,Art.,254)
and is therefore perpendicular to mn. (W.,Art.,197)
Therefore mab is the required right triangle.

VII. *An acute angle and the sum of the legs.*



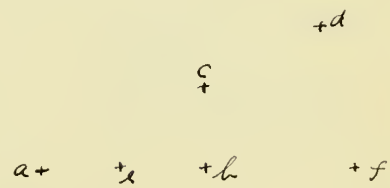
Let ae be the sum of the legs and ead the given acute angle.

To construct a right triangle with dae as an acute angle and ae as the sum of the legs.

At e construct ∠aec=45°. (Art.,16)
Find the intersection,c,of ad and ec. (Art.,7)
From c,draw cb perpendicular to ae. (Art.,12,II)
Then abc is the required triangle.

Proof: By construction,abc is a right triangle having the given acute angle bac.
Also bc=be and hence ab+bc=ab+be. (W.,Art.,147)
Therefore abc is the required triangle.

VIII. *An acute angle and the difference of the legs.*

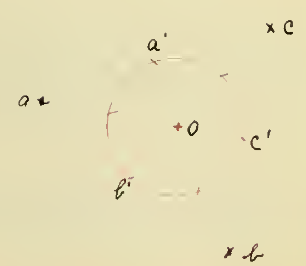


Let daf be the given acute angle and ae as the difference of the legs.

Construct ∠fec=45°. (Art.,16)
Find the intersection,c,of ad and ec. (Art.,7)
From c draw cb perpendicular to af. (Art.,12,II)
Then abc is the required triangle.

Proof: By construction,abc is a right triangle having the given acute angle daf.
Also be=bc and hence ae=ab-bc. (W.,Art.,147)
Therefore abc is the required triangle.

66. *To construct an equilateral triangle having given the radius of the inscribed circle.*



Let oa' be the given radius.

To construct an equilateral triangle of which oa' is the radius of the inscribed circle.

Describe a circle with oa' as a radius and in it inscribe an equilateral triangle a'b'c'. (Art.,63)

Draw tangents to the circle at a',b' and c' (Art.,26,I)
and find their intersections,a,b and c. (Art.,7)
Then abc is the required triangle.

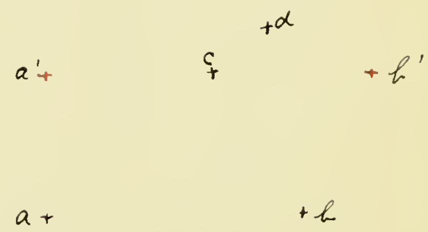
Proof: abc is circumscribed about the circle. Const.
abc is an equilateral triangle. (W.,Art.,440)
Therefore abc is the required triangle.

67. To construct a triangle having;

I. The base, the altitude, and the angle at the base.

Let ab be the given base,aa' the altitude and bad the given angle.

To construct a triangle having these given parts.



Draw a'b' parallel to ab and at a distance aa' from it, (Art.,20)
and find the intersection c of ad and a'b'. (Art.,7)
Then abc is the required triangle.

Proof: The triangle abc has the required base ab and the required angle bad. Const.
Its altitude is equal to the given altitude aa'. (W.,Art.,181)
Therefore abc is the required triangle.

II. The base, the altitude, and the angle at the vertex.

Let ab be the given base,cd the given altitude and acb the given angle at the vertex.

To construct a triangle having these given parts.

Upon ab describe a segment in which the given angle acb may be inscribed. (Art.,38)
Draw a'b' parallel to ab and at a distance cd from it. (Art.,20)
Find intersections c and c' with the arc. (Art.,4,II)
Then abc or abc' is the required triangle.

Proof; ab is the given base,cd the given altitude and acb or ac'b the given angle at the vertex. Therefore abc is the required triangle.

III. The base, the corresponding median and the angle at the vertex.

Let ab be the given base,oc the median and acb the given angle at the vertex.

To construct a triangle having these given parts.

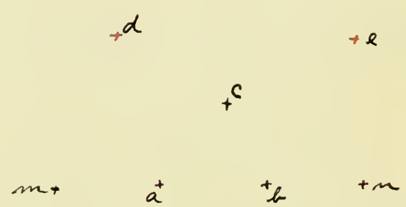
Upon ab describe a segment in which the given angle acb may be inscribed. (Art., 38)
Bisect ab at o. (Art., 14)
With o as a centre and the given median as a radius strike an arc locating c and c' .
Then abc or abc' is the required triangle.

Proof: By construction ab is the given base, oc the given median and acb the given angle at the vertex, Therefore abc or abc' is the required triangle..

IV. The perimeter and the angles.

Let mn be the given perimeter and φ, ψ and θ the given angles.

To construct a triangle whose perimeter is equal to mn and whose angles are equal to φ, ψ and θ.



Construct $\angle mne=\varphi/2$ and $\angle mnd=\psi/2$. (Art., 19)
Find the intersection c of me and nd (Art., 7)
Construct $\angle mca=\varphi/2$ and $\angle ncb=\psi/2$ (Art., 19)
Find the intersections a and b with mn (Art., 7)
Then abc is the required triangle.

Proof: ma=ac and nb=bc. (W., Art., 147)
Therefore ab+ac+bc=mn.
Also $\angle bac=\varphi/2+\varphi/2=\varphi$ and $\angle abc=\psi/2+\psi/2=\psi$. (W., Art., 137)
Therefore $\angle bca=\theta$. (W., Art., 132)
Therefore abc is the required triangle.

V. One side, an adjacent angle and the sum of the other sides.

Let ab be the given side, bac the given angle and ad the sum of the other two sides.

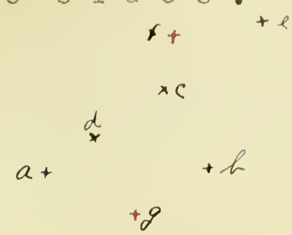
To construct a triangle having ab as one side, bac as one angle and ad as the sum of the other two sides.

Erect a perpendicular ef at the middle point of bd. (Art., 12, I)
Find the intersection c of ad and ef. (Art., 7)
Then abc is the required triangle.

Proof: bc=cd. (W., Art., 160)
Hence ac+cb=ac+cd . Therefore abc is the required triangle.

VI. One side, an adjacent angle and the difference of the other two sides.

Let ab be the given side, bae the given angle and ad as the difference of the other two sides.



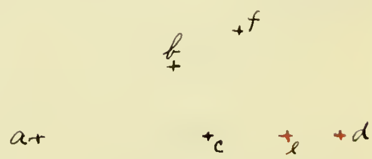
Construct fg perpendicular to bd at its middle point. (Art.,12,I)
Find the intersection c of ae and fg. (Art.,7)
Then abc is the required triangle.

Proof: bc=dc. (W.,Art.,169)
Hence ad=ac-bc
Therefore abc is the required triangle.

VII. The sum of the two sides and the angles.

Let af be the sum of the two given sides and φ, ψ and θ the given angles.

To construct a triangle whose angles are φ, ψ and θ and the sum of two of whose sides is equal to af.



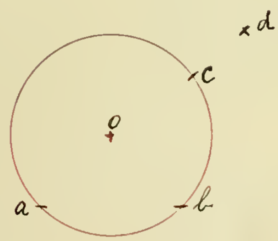
Construct $\angle fad=\varphi$ and $\angle afe=\psi$. (Art.,19)
Find the intersection e of fe and ad. (Art.,7)
Bisect $\angle afe$ by fc and find its intersection c with ad. (Art.,15)
Through c, draw cb parallel to ef. (Art.,20)
Find the intersection b of cb and af. (Art.,7)
Then abc is the required triangle.

Proof; $\angle bcf=\angle cfe=\angle bfc=\angle\psi/2$. (W.,Art.,110)
Hence bc=bf and ab+bc=ab+bf=af, (W.,Art.,147)
also $\angle abc=\angle afe=\angle\psi$, (W.,Art.,176)
and $\angle bac=\angle\varphi$, Const.
then $\angle acb=\theta$. (W.,Art.,132)
Therefore abc is the required triangle.

VIII. One side, an adjacent angle and the radius of the circumscribed circle.

Let ab be the given side, bad the given angle, and oa the radius of the circumscribed circle.

To construct a triangle having these given parts.

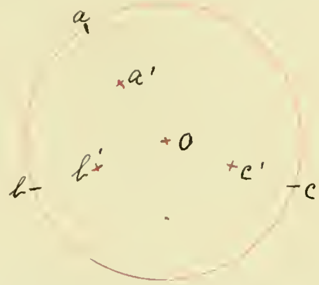


With a and b as centres and oa as a radius, describe arcs intersecting at o. With o as a centre and oa as a radius, describe a circle

and find its intersection c with ad. (Art.,4,II)
Then abc is the required triangle.

Proof: By construction it has the given side ab, the given angle bac and the given circumscribed circle. Therefore it is the required triangle.

IX. *The angles and the radius of the circumscribed circle.*



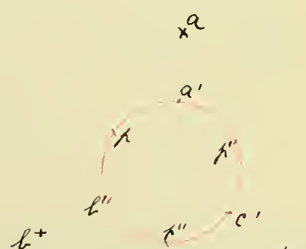
Let a', b' and c' be the given angles and oa the radius of the circumscribed circle.

To construct a triangle having these givne angles and inscribed in the given circle.

Construct the triangles a'b'c' having the given angles and find the centre o of its circumscribed circle. (Art.,34)
With o as a centre and oa as a radius, describe a circle.
Find the intersections a, b and c of oa', ob' and oc' with the circumference. (Art.,4,I)
Then abc is the required triangle.

Proof: abc is inscribed in the given circle. Const.
ab is parallel to a'b', bc to b'c' and ac to a'c'. (W.,Art.,345)
Hence $\angle a=\angle a'$, $\angle b=\angle b'$ and $\angle c=\angle c'$. (W.,Art.,176)
Therefore abc is the required triangle.

X. *The angles and the radius of the inscribed circle.*



Let a', b' and c' be the given angles and op the radius of the inscribed circle.

To construct a triangle having these given angles and circumscribed about the given circle.

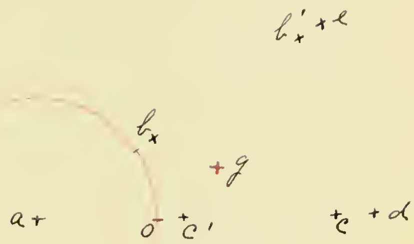
Construct any triangle a'b'c' having the given angles and find the centre o of its inscribed circle. (Art.,35)
With o as a centre and op as a radius, describe a circle.
Draw op perpendicular to a'b', op' to a'c' and op" to b'c'. (Art.,12,II)
Find the intersections p, p' and p". (Art.,36,I)
Draw tangents to the circle at p, p' and p". (Art.,36,I)
Find their intersections a, b and c. (Art.,7)
Then abc is the required triangle

Proof: abc is circumscribed about the circle. Const.
op is perpendicular to ab, op' to ac and op" to bc. (W.,Art.,254)
Then ab is parallel to a'b', ac to a'c', bc to b'c'. (W.,Art.,104)
Hence $\angle a=\angle a'$, $\angle b=\angle b'$ and $\angle c=\angle c'$. (W.,Art.,176)

Therefore abc is the required triangle.

XI. An angle, and the bisector and the altitude drawn from the vertex of the given angle.

Let ead be the given angle, ag its bisector and ao the given altitude.



To construct a triangle having these given parts.

Construct the bisector ag of the angle ead. (Art.,15)

Lay it off equal to the given bisector. (Art.,4,Cor.)

With a as a centre and the given altitude as a radius, describe a circle. From g draw tangents gb and gc' to the circle. (Art.,36,II) and find the intersections b,c,b' and c' of these tangents with ae and ad. (Art.,7)

Then abc or ab'c' is the required triangle.

Proof: By construction, either of the triangles abc or ab'c' has the given angle and the given bisector. The altitude is the perpendicular froma to bc or b'c' which in either case is ao, the given altitude. (W.,Art.,253)

Therefore abc or ab'c' is the required triangle.

XII. Two sides and the median corresponding to the other side.

Let ab and ac be the two given sides and ad the given median.

To construct a triangle having ab and ac as two of its sides and ad the median to the third side.

Bisect ab at f. (Art.,15)

With f and a as centres and radii 1/2 ac and ad respectively, strike arcs intersecting at d. With d and a as centres and radii af and df respectively, strike arcs intersecting at e. Multiply ae by two, locating the point c. (Art.,21)

Then abc is the required triangle.

Proof: The triangle abc has the given sides ab and ac. Const.

The quadrilateral aedf is a parallelogram. (W.,Art.,182)

Then ed is parallel to ab and equal to one-half of it.

By construction e is the middle point of ac.

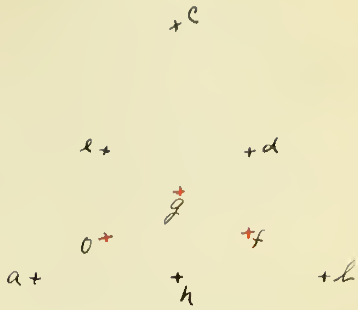
Hence d is the middle point of bc. (W.,Art.,188)

Therefore abc is the required triangle.

XIII. The three medians.

Let ad, be and cb be the given medians.

To construct a triangle having these given medians.



Divide ad into three equal parts at o and g. (Art., 22)

With g and d as centres and radii equal re-spectively to 1/3 be and 1/3 ch, describe arcs locating the point f. Extend fg to e and b making eg=fb=1/3 eb. (Art., 4, Cor,)

Extend the lines ae and bd. (Art., 4, Cor.)

Find their intersection c. (Art., 7)

Then abc is the required triangle.

Proof: ed=of=1/2 ab and the lines are parallel. (W., Art., 189)

Hence edfo is a parallelogram. (W., Art., 183)

Find the intersections h and r of the median from c with ab and of,

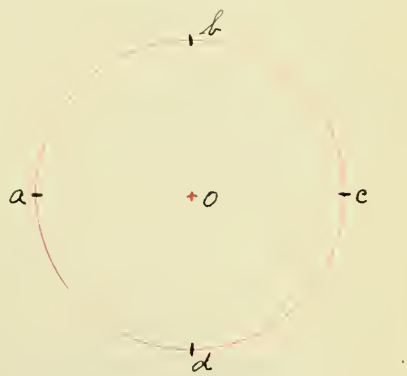
Then gr=1/2 df=1/2 gh. Hence df=gh and since df=1/3 ch we have gh=1/3 ch and therefore ch is the given median. Therefore the triangle has the three given medians and must be the required triangle.

68. To construct a square, having given;

I. The diagonal.

Let ac be the given diagonal.

To construct a square having ac as a diagonal.



Construct a circle on ac as a diameter. (Art., 14)

Erect a perpendicular to ac at o. (Art., 12, I)

Find its intersection b and d with the circumference. (Art., 4, I)

Then abc is the required square.

Proof: Since the angles at o were constructed equal, then the arcs ab, bc, cd and da are equal and therefore the chords ab, bc, cd and da are equal. (W., Art., 236, 241)

Also $\angle a=\angle b=\angle c=\angle d=$ a right angle. (W., Art., 290)

Therefore abcd is the required square.

II. The sum of the diagonal and one side.

Let fc be the sum of the diagonal and one side.

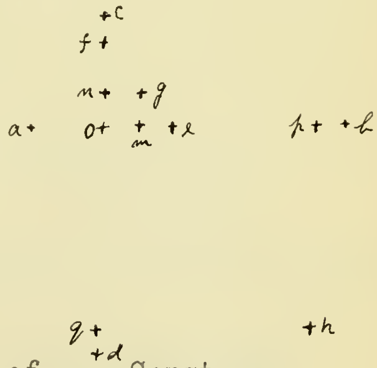
To construct a square such that the sum of its side and diagonal shall be equal to fc.

At c construct $\angle fce=45^\circ$, (Art.,16)
and at f construct $\angle cfd=1/2\ \angle fce$. (Arts.,15,19)
Find the intersection, d, of fd and ce. (Art.,7)
With d as a centre and dc as a radius, describe an arc cutting fc at a. (Art.,4,II)
With a and c as centres and the same radius, describe arcs intersecting at b. Then abcd is the required square.

Proof: ab=bc=cd=da.	Const.
Since ad=dc then $\angle dac=\angle acd=45^\circ$.	(W.,Art.,145)
Therefore $\angle adf=22\ 1/2^\circ$.	(W.,Art.,137)
Therefore af=ad.	(W.,Art.,147)
Since $\angle dac=\angle acd=45^\circ$ then $\angle adc=90^\circ$.	(W.,Art.,129)

Similarly for the other angles. Therefore abcd is the required square.

69. Given two perpendiculars, ab and cd, intersecting in o, and a straight line intersecting these perpendiculars in e and f; to construct a square, one of whose angles shall coincide with one of the right angles at o, and the vertex of the opposite angle of the square shall lie in ef. (Two solutions.)

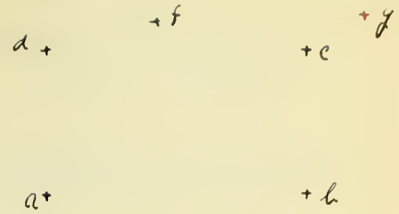


Bisect the angles boc and bod (Art.,15) and find the intersections g and h of these bisectors with ef. (Art.,7)
From g and h construct perpendiculars to ab and cd. (Art.,12,MI)
And find their intersections n, m, p and q with these lines. (Art.,7)
Then mn or pq is the required square.
Proof: Each square has a vertex at o and one in ef. Const.
The angles of nm and pq are right angles, Const.
and also ng=mg and ph=gh. (W.,Art.,162)
Therefore nm and pq are squares and either would be the required square.

70. To construct a rectangle, having given;

I, One side and the angle between the diagonals.

Let ab be the given side and eaf the angle between the diagonals.
To construct a rectangle having ab as one side and eaf the angle between the diagonals.



Extend ea to b making ab equal to the given side. (Art.,4,Cor.)
Construct the bisector ag of the angle baf. (Art.,15)
At b, construct bc perpendicular to ab. (Art.,12,II)
Find the intersection c of ag and bc. (Art.,7)
Construct ad perpendicular to ab and cd perpendicular to bc and find their intersection d. Then abcd is the required rectangle.

Proof: By construction the angles are right angles and therefore abcd is a rectangle.
The angle between the diagonals=$180°-(\angle cab+\angle abd)$ (W.,Art.,129)
or $=180°-2\angle cob=180°-\angle fab=\angle eaf$.
Therefore abcd is the required rectangle.

II. The perimeter and the diagonal.

Let am be the given perimeter and ac the given diagonal.
To construct a rectangle with the perimeter equal to am and diagonal ac.
Bisect am at n. (Art.,14)
Construct angle anp=45°. (Art.,16)
With a as a centre and ac as a radius, describe a circle and find its intersection c with np. (Art.,4,II)
Construct cb perpendicular to an, cd to cb and ad to ab and find the intersections b and d. (Art.,12,7)
Then abcd is the required rectangle.

Proof: By construction it has the given diagonal ac and since the angles are all right angles it is a rectangle.
Angle bcn=45°. (W.,Art.,129)
Therefore bc=bn. (W.,Art.,147)
Therefore ab+bc=an=nm. Hence ab+bc+cd+ad=am.
Therefore abcd is the required rectangle.

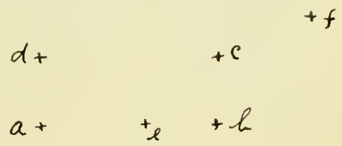
III. The perimeter and the angle between the diagonals.

Let am be the given perimeter and φ as the angle between the diagonals.
Bisect am at n. (Art.,14)
Construct angle anp=45°. (Art.,16)
At a construct angle cab=$1/2(180°-\varphi)$. (Art.,19)

Find the intersection c of ac and np. (Art.,7)
Construct ad and bc perpendicular to ab and cd perpendicular to bc,. and find the intersections b and d. (Arts.,12,7)
Then abcd is the required rectangle.

Proof: abcd is a rectangle and has the given perimeter by the same proof as the last article.
The angle, φ, between the diagonals $=180°-(\angle cab+\angle abd)$. (W.,Art.,129)
Therefore $\varphi = 180°-2[1/2(180°-\varphi)]=\varphi$
Therefore abcd is the required rectangle.

IV. The difference between two adjacent sides and the angle between the diagonals.



Let ae be the difference between two adjacent sides and φ the angle between the diagonals.

To construct a rectangle having these given parts.

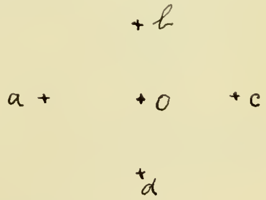
At a construct $\angle fab=1/2(180°-\varphi)$. (Art.,19)
At e construct $\angle ceb=45°$. (Art.,16)
Find the intersection c of af and ec. (Art.,7)
Construct cb and ad perpendicular to ab and cd perpendicular to bc and find the intersections b and d. (Arts.,12,7)
Then abcd is the required rectangle.

Proof: abcd is a rectangle and has the required angle between the diagonals by the same proof as in the last article.

$\angle bce=\angle bec=45°$. (W.,Art.,129)
Therefore be=bc. (W.,Art.,147)
Hence ae=ab-bc.
Therefore abcd is the required rectangle.

71. To construct a rhombus, having given;

I. The two diagonals.



Let ac and bd be the given diagonals.

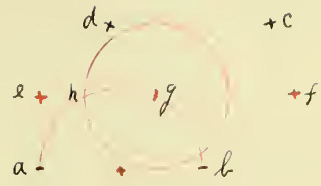
To construct a rhombus with diagonals ac and bd.

Construct a perpendicular at the middle point of the diagonal ac. (Art.,12,I)
Lay off ob=od=1/2 the given diagonal bd. (Art.,4,I)
Then abcd is the required rhombus.

Proof: The quadrilateral has the given diagonals Const.
and also ab=bc=cd=ad. (W.,Art.,160)
Therefore abcd is the required rhombus.

II. One side and the radius of the inscribed circle.

Let ab be the given side and ae the radius of the inscribed circle.



To construct a rhombus having these given parts.

Construct a semicircle on ab as a diameter. (Art.,14)

Construct ef parallel to ab and at a distance ae from it. (Art.,20)

Find the intersections g and h of ef with the circumference. (Art.,4,II)

With either g or h as a centre and ae as a radius, describe a circle.

From a and b draw tangents to the circle and lay them off equal to ab. (Arts.,36; 4,Cor.)

Then abcd is the required rhombus.

Proof: It has the given side and inscribed circle. Const.

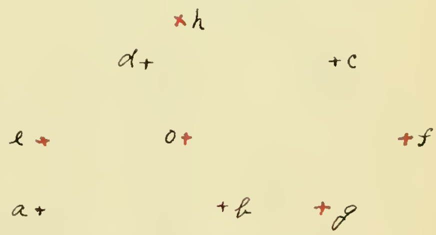
Since the diagonals ac and bd bisect each other at right angles, (W.,Art.,290)

then ab=ad=bc=dc. (W.,Art.,160)

Therefore abcd is a rhombus and is the required rhombus.

III. One angle and the radius of the inscribed circle.

Let gah be the given angle and ae the radius of the inscribed circle.



To construct a rhombus having gah as one angle and ae as the radius of the inscribed circle.

Construct the bisector ac of the angle gah. (Art.,15)

Construct ef parallel to ab and at a distance ae from it. (Art.,20)

Find the intersection o of ef and ac. (Art.,7)

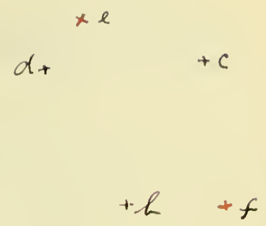
Lay off oc=oa. (Art.,4,Cor.)

Construct cd parallel to ag and cb parallel to ah and find the intersections d and b with ah and ag. (Art.,20,7)

Then abcd is the required rhombus.

Proof: By construction abcd is a parallelogram and since the diagonal ac bisects the angles (which are not right angles) it must be a rhombus. Since o is the middle of the diagonal and at a distance ae from ab it is the same distance from all the sides. Therefore this rhombus has the given angle and inscribed circle and must be the required rhombus.

IV. One angle and one of the diagonals.



Let eaf be the given angle and ac the given diagonal.

To construct a rhombus having eaf as an angle and ac as a diagonal.

Construct the bisector of the angle eaf and on it lay off the given diagonal ac. (Arts.,15;4,Cor.)

Construct cd parallel to af and cb parallel to ae and find their intersections d and b with ae and af. (Art.,20,7)

Then abcd is the required rhombus.

Proof; By construction, abcd is a parallelogram and since the diagonal ac bisects the angles it must be a rhombus. And since it has the given angle and diagonal it must be the required rhombus.

72. To construct a rhomboid, having given:

I. One side and the two diagonals.

Let ab be the given side and ac and bd the given diagonals.

To construct a rhomboid with ab as a side and ac and bd as diagonals.

Bisect the diagonals ac and bd. (Art.,14)

With a and b as centres and these half diagonals as radii, describe arcs intersecting at o.

Extend bo to d and ao to c making od=ob and oc=ao. (Art.,4,Cor.)

Then abcd is the required rhomboid.

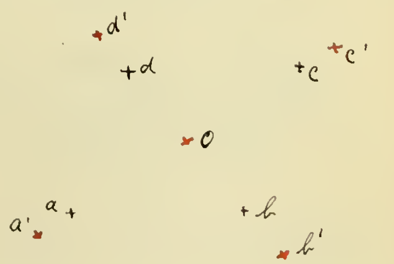
Proof: From the four triangles whose vertices are at o, we have

dc=ab and ad=bc. (W.,Art.,143)

Hence abcd is a parallelogram. (W.,Art.,182)

Since the angles are not right angles, it is a rhomboid and must be the required rhomboid.

II. The diagonals and the angle between them.



Let ac and bd be the given diagonals and aob the given angle.

To construct a rhomboid having ac and bd as diagonals and aob the angle between the diagonals.

Construct ∠a'ob' equal to the given

angle. (Art.,19)

Extend a'o to c' and b'o to d'. (Art.,4,Cor.)

Bisect the given diagonals ac and bd. (Art.,14)

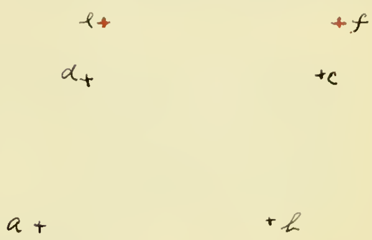
Lay off od=ob=1/2 bd and oa=oc=1/2ac. (Art.,4,Cor.)

Then abcd is the required rhomboid.

Proof: The quadrilateral abcd has the given diagonals and the angle between them. Const.

From the triangles as in the last article we have that abcd is a rhomboid. Therefore it is the required rhomboid.

III. One side, one angle, and one diagonal.



Let ab be the given side, eab the given angle and ac the given diagonal.

To construct a rhomboid having these given parts.

Construct bf parallel to ae. (Art.,20)

With a as a centre and the given diagonal as a radius, describe an arc and find its intersection c with bf. (Art.,41)

Construct cd parallel to ab and find its intersection d with ae. (Arts.,20,7)

Then abcd is the required rhomboid.

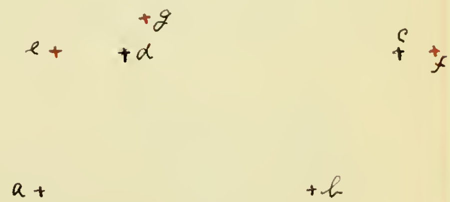
Proof: The quadrilateral has the given parts. Const.

ab is parallel to dc and bc is parallel to ad. Const.

Therefore abcd is a rhomboid. (W.,Art.,169)

And must be the required rhomboid.

IV. The base, the altitude, and one angle.



Let ab be the given base, ae the altitude and gab the given angle.

To construct a rhomboid having these given parts.

Through e, construct ef parallel to ab and find the intersection d of ag and ef. (Arts.,29,7)

Construct a parallelogram abcd having the two sides ab and ad and the included angle bad. (Art.,33)

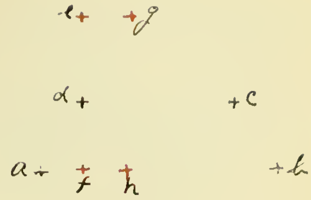
Then abcd is the required rhomboid.

Proof: abcd is a parallelogram having the given parts, Const. and since the angles are not right angles it is the required rhomboid. (W.,Art.,169)

73. To construct an isosceles trapezoid having given:-

I. The bases and one angle.

Let ab and bh be the given bases and gab the given angle.



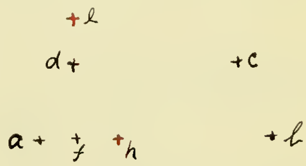
To construct a trapezoid having ab and bh as bases and gab as one of the angles.

Construct a perpendicular, ef, at the middle point of ah. (Art., 12, I)
Find the intersection, d, of ag and ef. (Art., 7)
Construct dc parallel to ab
and lay it off equal to bh. (Arts., 20; 4, Cor.)
Then abcd is the required trapezoid.

Proof: By construction ab and cd are parallel, therefore abcd is a trapezoid. (W., Art., 165)
Since ad=dh (W., Art., 160)
and also dh=cb (W., Art., 178)
then ad=cb and therefore abcd is the required trapezoid.

II. The bases and the altitude.

Let ab and bh be the given bases and df the given altitude.



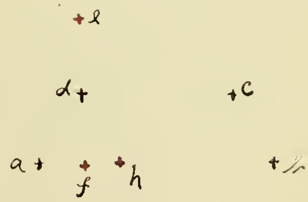
To construct an isosceles trapezoid, having the given bases and altitude.

Construct a perpendicular, ef at the middle point of ah. (Art., 12, I)
Lay off fd equal to the given altitude. (Art., 4, Cor.)
Construct dc parallel to ab
and lay it off equal to bh. (Arts., 20; 4, Cor.)
Then abcd is the required trapezoid.

Proof: By construction, it is a trapezoid having the given bases and altitude.
Since ad=dh (W., Art., 160)
and also dh=cb (W., Art., 178)
then ad=bc and therefore the trapezoid is isosceles and is the required trapezoid.

III. The bases and the diagonal.

Let ab and bh be the given bases and bd the given diagonal.



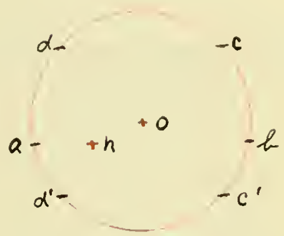
To construct an isosceles trapezoid having ab and bh as bases and bd as a diagonal.

Construct a perpendicular, ef, at the middle point of ah. (Art., 12, I)
With b as a centre and bd as a radius,

describe an arc and find its intersection d with ef. (Art.,4,II)
Construct dc parallel to ab
and lay it off equal to bh. (Arts.,20;4,Cor)
Then abcd is the required trapezoid.

Proof: By construction,abcd has the given parts and the proof that it is an isosceles trapezoid is the same as that of the last article.

IV. The bases and the radius of the circumscribed circle.



Let ab and bh be the given bases and oa the radius of the circumscribed circle.

To construct an isosceles trapezoid having the given bases and the given circumscribed circle..

Construct a perpendicular dd' at the middle point of ah. Art.,12,I)
Find its intersections d and d' with the circle. (Art.,4,II)
Construct dc and d'c' parallel to ab
and find their intersections c and c' with the circle. (Arts.,20;4)
Then abcd or abc'd' is the required trapezoid.

Proof: By construction,abcd or abc'd' has the required parts and since ad=bc and ad'=bc', (W.,Arts,,257,241)
either abcd or abc'd' is the required isosceles trapezoid.

74. To construct a trapezoid, having given:

I. The four sides.

Let ab,bc,cd and ad be the given sides.

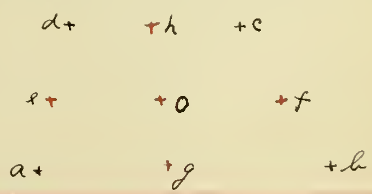
To construct a trapezoid having these given sides.

Lay off ae equal to the given side dc. (Art.,4,Cor.)

With e and b as centres and radii ad and bc respectively strike arcs intersecting in c.
Construct the parallelogram aecd. (Art.,33)
Then abcd is the required trapezoid.

Proof: Since dc=ae and ad=ec (W.,Art.,178)
then the quadrilateral abcd has the given sides. And since de is parallel to ab it is a trapezoid. Therefore abcd is the required trapezoid.

II. The two bases and the two diagonals.



Let ab and cd be the given bases and ac and bd the given diagonals.

To construct a trapezoid having these

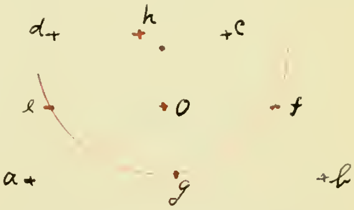
given parts.

Construct the line ef=1/2(ab+cd). (Art.,5,14)
With e and f as centres and radii 1/2 bd and 1/2 ac respectively locate the point g. With e and f as centres and radii 1/2 ac and 1/2 bd respectively locate the point h. Then egfh is a parallelogram. (W.,Art.,182)
Through g and h, construct ab and cd respectively parallel to ef and make ag=gb=1/2 ab and dh=ch=1/2 cd. (Art.,20;4,Cor.)
Then abcd is the required trapezoid.

Proof: By construction, abcd is a trapezoid having the given bases.
o is the middle point of hg. (W.,Art.,184)
of is parallel to gb and equal to 1/2(hc+gb). Const.
Therefore f is the middle point of bc. (W.,Art.,188)
Hence gf=1/2 ac. (W.,Art.,189)
Since gf was constructed as half of the given diagonal, then ac must be the given diagonal. Similarly bd is the other given diagonal. Therefore abcd is the required trapezoid.

III. The bases, one diagonal, and the angle formed by the diagonals.

Let ab and cd be the given bases, ac the diagonal and φ the angle between the diagonals.



To construct a trapezoid having these given parts.

Construct the line ef=1/2(ab+cd). (Arts.,5,14)
On the line ef construct the segment of a circle in which the angle φ may be inscribed. (Art.,38)
With f as a centre and 1/2 ac as a radius, describe an arc cutting the arc of the segment at g.
Construct the parallelogram egfh. (Art.,33)
The construction from here on is the same as in the last article.
Then abcd is the required trapezoid.

Proof: Since eg is parallel to bd and gf parallel to ac then the angle between the diagonals is equal to the angle egf=φ. The trapezoid abcd also has the given bases and diagonal and is therefore the required trapezoid.

75. To construct a circle which has the radius r and which also;

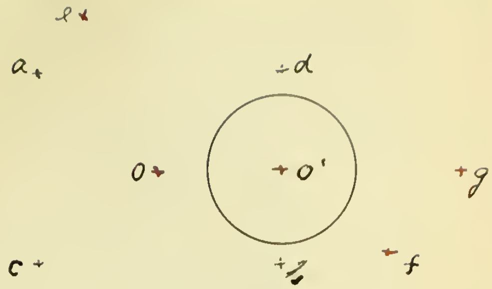
I. Touches each of two intersecting lines ab and cd.

Let ab and cd be the given lines and r the given radius.

To construct a circle of radius r

tangent to ab and cd.

Find the intersection o of ab and cd. (Art.,7)



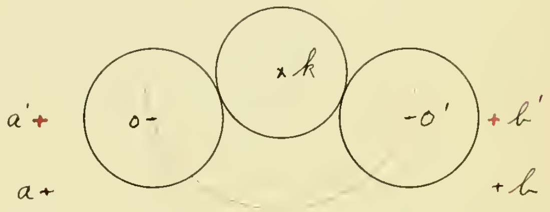
Construct the bisector og of the angle bod. (Art.,15)
Construct ef parallel to ab and at the distance r from it. (Art.,20)
Find the intersection o' of ef and og. (Art.,7)
With o' as a centre and r as a radius, describe the given circle.

Proof: o' is equidistant from od and ob. (W.,Art.,162)
It is the distance r from ob and therefore also from od. (W.,Art.,181)
Hence a circle described with o' as a centre and r as a radius will be tangent to ab and cd and is the required circle.

II. T o u c h e s a g i v e n l i n e ab a n d a g i v e n c i r c l e k.

Let ab be the given line and k the given circle.

To construct a circle of radius r which shall be tangent to ab and also to k.



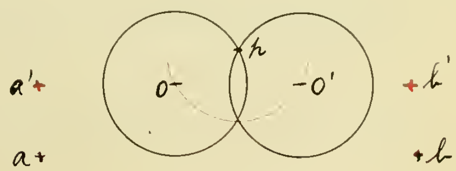
Find the sum of r and the radius of the circle k. (Art.,5)
With this as a radius describe a circle.
Construct a'b' parallel to ab and at a distance r from it. (Art.,20)
Find the intersections o and o' of a'b' with the circumference. (Art.,4,II)
With o and o' as centres and r as a radius describe circles.
Then either of these is the required circle.

Proof: By construction, o and o' are at the distance r from ab and r plus the radius of k from k. Therefore circles described with o and o' as centres and r as a radius will be tangent to ab and to the circle k.

III. P a s s e s t h r o u g h a g i v e n p o i n t p a n d t o u c h e s a g i v e n l i n e ab.

Let ab be the given line and p the given point..

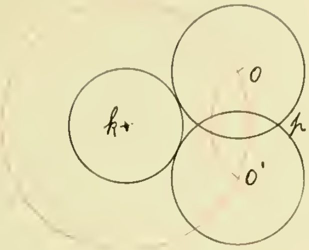
To construct a circle of radius r, tangent to ab and passing through p.



Construct a'b' parallel to ab and at a distance r from it. (Art.,20)
With p as a centre and radius r, describe a circle and find its intersections o and o' with a'b'. (Art.,4,II)
With o and k' as centres and r as a radius, describe circles.
Then either of these is the required circle.

Proof: Both o and o' are at the distance r from p and ab. Const. Therefore a circle described with o or o' as a centre and r as a radius will pass through p and be tangent to ab.

IV. Passes through a given point p and touches a given circle k.



Let k be the given circle and p the given point.

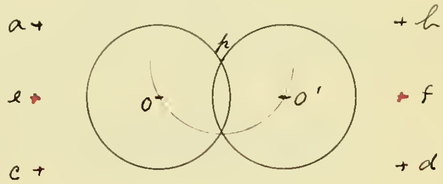
To construct a circle of radius r passing through the point p and tangent to k.

Construct r plus the radius of the circle k. (Art.,5)
With this radius and k as a centre describe a circle. With p as a centre and r as a radius describe a circle cutting this circumference at o and o'. Then with o or o' as a centre and r as a radius describe the required circle.

Proof: By construction, both o and o' are at the distance r from p and the circumference of k. Then a circle described with either o or o' as a centre and r as a radius will pass through p and be tangent to k.

76. To construct a circle which shall;

I. Touch two given parallels and pass through a given point p.



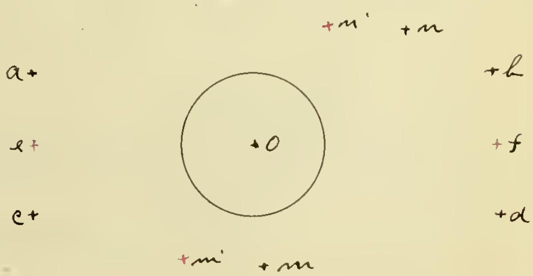
Let ab and cd be the given parallels and p the given point.

To construct a circle passing through p and tangent to both ab and cd.

Bisect ac at e. (Art.,14)
Construct ef parallel to ab. (Art.,20)
With p as a centre and ae as a radius, describe a circle, and find its intersections, o and o', with ef. (Art.,4,II)
With o or o' as a centre and ae as a radius describe the required circle.

Proof: By construction, either o or o' is at the distance ae from p and the two parallels ab and cd. Then if we take o or o' as centre and ae as radius and describe a circle, it will pass through p and be tangent to both ab and cd.

II. Touch three given lines two of which are parallel.



Let ab and cd be the given parallels and mn the other given line.

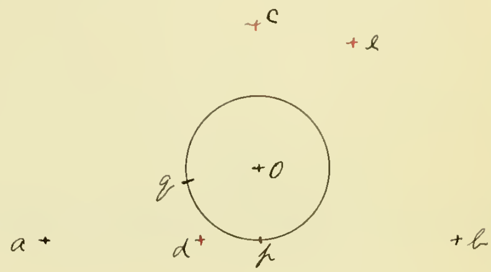
To construct a circle tangent to ab, cd and mn.

Bisect the perpendicular distance ac

at e. (Art.,14)
Through e construct ef parallel to ab. (Art.,20)
Construct m'n' parallel to mn and at a distance ae from it. (Art.,20)
Find the intersection o of ef and m'n'. (Art.,7)
With o as a centre and ae as a radius, describe a circle. Then this is the required circle.

Proof: Since o lies in ef it is the distance ae from ab and cd and since it lies in m'n' it is the distance ae from mn. Therefore if we describe a circle with o as a centre and ae as a radius it will be tangent to the three lines ab, cd and mn.

III. T o u c h a g i v e n l i n e ab a t p a n d p a s s t h r o u g h a g i v e n p o i n t g.



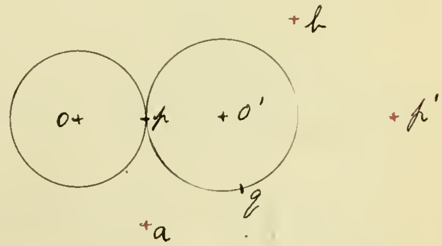
Let p be some point on the given line ab and q some other given point.

To construct a circle tangent to ab at p and passing through q.

At p, construct pc perpendicular to ab. (Art.,12,I)
Also construct de perpendicular to pq at its middle point. (Art.,12,I)
Find the intersection o of pc and de. (Art.,7)
With o as a centre and op as a radius, describe a circle. Then this is the required circle.

Proof: Since o lies on de, it is equidistant from p and q. (W., Art.,160)
Therefore the circle passes through q.
Since op is perpendicular to ab the circle is tangent to ab at p. (W., Art.,253)
Therefore this is the required circle.

IV. T o u c h a g i v e n c i r c l e a t p a n d p a s s t h r o u g h a g i v e n p o i n t q.



Let p be the given point on the circle o and q any other point.

To construct a circle passing through q and tangent to the circle o at the point p.

Extend the line op to p'. (Art.,4,Cor)
Construct ab perpendicular to pq at its middle point. (Art.,12,I)
Find the intersection o' of pp' and ab. (Art.,7)
With o' as a centre and o'p as a radius, describe a circle.

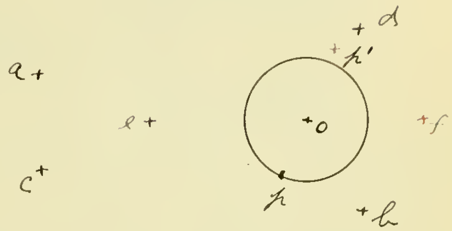
Then this is the required circle.

Proof: Since c' lies on ab it is equidistant from p and q. (W.,Art.,160)

Then if we take o' as a centre and o'p as a radius and describe a circle it will pass through q.

V. Touch two given lines and touch one of them at a given point p.

Let ab and cd be the given lines and p the point on the line ab.



To construct a circle tangent to cd and tangent to ab at the point p.

Find the intersection e of ab and cd. (Art.,7)

Construct the bisector ef of the angle bed. (Art.,15)

Erect a perpendicular pp' to ab at the point p. (Art.,12,I)

Find the intersection o of ef and pp' (Art.,7)

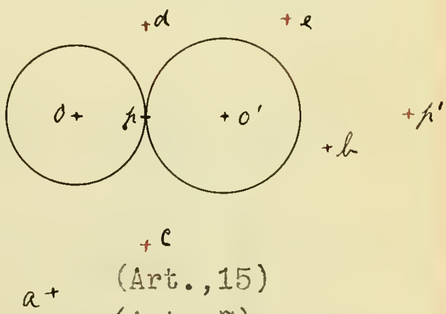
With o as a centre and op as a radius, describe a circle. Then this is the required circle.

Proof: The point o is equidistant from ab and cd. (W.,Art.,162)

Therefore a circle constructed with o as a centre and op as a radius will be tangent to ab and cd.

VI. Touch a given line and touch a given circle at a point p.

Let ab be the given line and p the point on the given circle o.



To construct a circle tangent to ab and to the given circle at the point p.

Extend op to p'. (Art.,4,Cor)

Construct a perpendicular ad to pp' at p and find its intersection c with ab. (Arts.,12;7)

Construct the bisector ec of the angle dcb. (Art.,15)

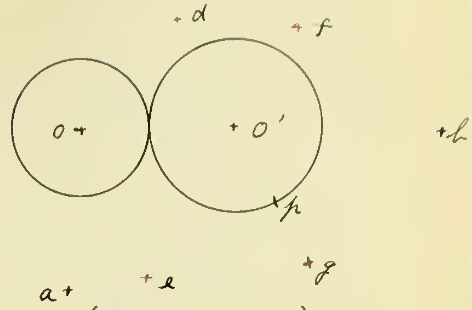
Find the intersection o' of ec and pp'. (Art.,7)

With o' as a centre and o'p as a radius, describe a circle. Then this is the required circle.

Proof: Since o' lies on ce it is equidistant from ab and cd. (W., Art.,162)

Therefore a circle described with o' as a centre and o'p as a radius will be tangent to ab and also to cd at the point p. It must, therefore, be tangent to the given circle at the point p.

VII. Touch a given line ab at p and also touch a given circle.



Let p be the given point on the line ab and o the given circle.

To construct a circle tangent to the given circle and tangent to ab at p.

Construct gd perpendicular to ab at p. (Art.,12,I)

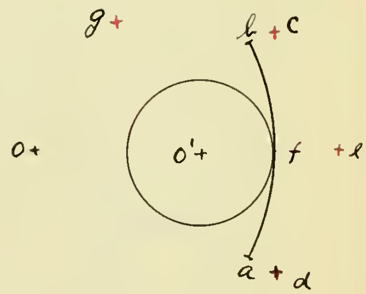
Lay off pg equal to the radius of o. (Art.,4,Cor.)

Construct a perpendicular ef at the middle point of og and find its intersection o' with gd. (Arts.,12;7)

With o' as a centre and o'p as a radius, describe a circle. Then this is the required circle.

Proof: The point o' is equidistant from o and g. (W.,Art.,160) Subtracting the radius of o from each of these distances we get the distance to the circumference of o equal to o'p. Therefore a circle described with o' as a centre and o'p as a radius will be tangent to ab and also to the given circle.

77. To inscribe a circle in a given sector.



Let aob be the given sector.

To inscribe a circle in aob.

Construct the bisector oe of the angle aob. (Art.,15)

Find the point f where oe cuts the arc ab. (Art.,4,I)

Construct cd perpendicular to oe at the point f and find its intersections c and d with ob and oa. (Arts.,12;7)

Construct the bisector dg of the angle odc. (Art.,15)

Find the intersection o' of dg and ce. (Art.,7)

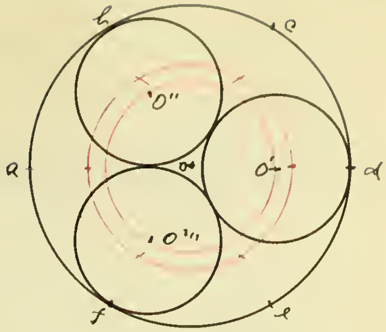
With o' as a centre and o'f as a radius, describe a circle. Then this is the required circle.

Proof: The circle o' is inscribed in the triangle ocd. (Art.,35) Since it passes through the point f which is also a point on the arc ab it is also inscribed in the sector aob.

78. To construct within a given circle three equal circles so that each will touch the other and also the given circle.

Let o be the given circle.

To construct three circles within o which shall be tangent to it and to each other.



Use the radius as a chord and divide the circumference into six equal parts. Inscribe a circle o' in the sector eoc. (Art.,77)

With o as a centre and oo' as a radius, describe a circle and find its intersections o" and o"' with ob and of. (Art.,4,I)

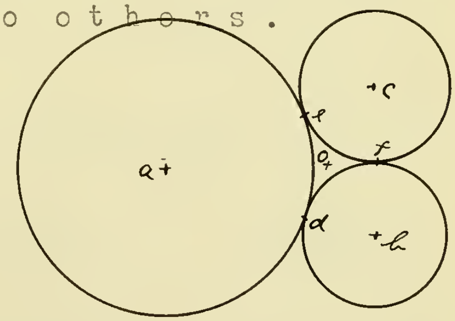
With o" and o"' as centres and o'd as a radius describe circles. Then o',o" and o"' are the required circles..

Proof: Since the two circles o' and o" are inscribed in the sectors eoc and coa they are both tangent to oc. Let p and p' be their points of tangency. Then in the two right triangles oo'p and oo"p' ,oo'=oo" and o"p'=o'p Const.

Hence the triangles are equal (W.,Art.,151)

and op=op'. Therefore the points p and p' coincide and the circles are tangent to each other. Similarly for all the other circles. Therefore, o',o" and o"' are the required circles.

79. To describe circles about the vertices of a given triangle as centres so that each shall touch the two others.



Let abc be the given triangle.

To describe circles with a,b and c as centres, each of which shall touch the other two.

Find the intersection o of the bisectors of the angles a,b and c. (Art.,35)

Construct perpendiculars od,oe and of to the sides of the triangle. (Art.,12,II)

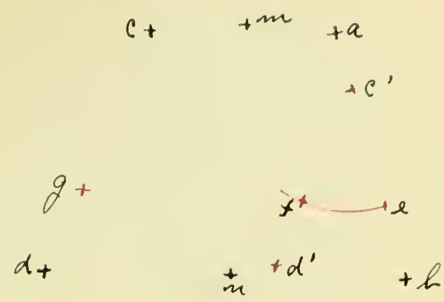
Then tri.aoe=tri.aod,tri.coe=tri.cof and tri.bof=tri.bod.(W.,Art.,141)

Therefore ae=ad,ce=cf and bd=bf. With a,b,and c as centres and radii ae,bd and cf respectively, describe three circles. Then these are the required circles.

Proof: Since by construction the distance between the centres of the circles is equal to the sum of their radii, each circle must be tangent to the other two. Therefore these are the required circles.

80. To bisect the angles formed by two

lines, without producing the lines to their point of intersection.



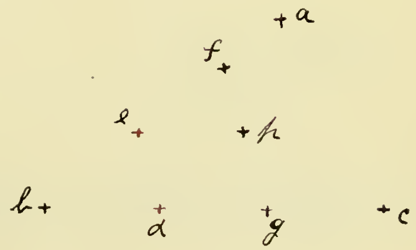
Let ab and cd be the given lines.

To bisect the angle between ab and cd.

From any point c' of ab construct c'd' parallel to cd. (Art.,20)
Lay off c'e=c'f, extend ef to g and find its intersection g with cd. (Arts.,4,Cor;7)
Construct a perpendicular mn at the middle point of eg. (Art.,12,D)
Then mn is the required bisector.

Proof: ∠cge=∠afe=∠aeg (W.,Art.,176)
Therefore g,e and the point of intersection of ab and cd form an isosceles triangle. (W.,Art.,147)
Therefore the perpendicular bisector of eg bisects the angle between ab and cd. (W.,Art.,149)

81. To draw through a given point p between the sides of an angle abc a line terminated by the sides of the angle and bisected at p.



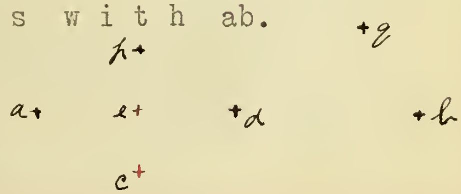
Let abc be the given angle and p the given point.

To draw a line through p included between ab and bc which is bisected at p.

Construct pe parallel to bc and pd parallel to ab. (Art.,209)
Find their intersections d and e with bc and ab. (Art.,7)
Lay off ef=pd and dg=ep. (Art.,4,Cor.)
Then fg is the required line.

Proof: Triangle fpe=triangle pdg. (W.,Art.,143)
Therefore pf=pg.
Since the lines ep and bc are parallel and ∠fpe=∠hgd then fpg must be a straight line. Therefore it is the required line.

82. Given two points p,q and a line ab; to draw lines from p and q which shall meet on ab and make equal angles with ab.



Let ab be the given line and p and q the given points.

To draw lines from p and q meeting on ab

and making equal angles with ab.

Construct pc perpendicular to ab. (Art.,12,II)

Find the intersection e of pc and ab. (Art.,7)

Lay off pe=ec. (Art.,4,Cor.)

Find the intersection d of ab and cq. (Art.,7)

Then pd and dq are the required lines.

Proof: $\angle pda=\angle adc$, (W.,Art.,95)

and $\angle qdb=\angle adc$. (W.,Art.,93)

Hence $\angle pda=\angle qdb$.

Therefore pd and qd are the required lines.

83 To find the shortest path from p to q which shall touch a line ab.

Let ab be the given line and p and q the given points.

To find the shortest path from p to q which shall touch ab.

Construct pp' perpendicular to ab. (Art.,12,II)

Find the intersection, e, of ab and pp'. (Art.,7)

Lay off ep'=ep. (Art.,4,Cor.)

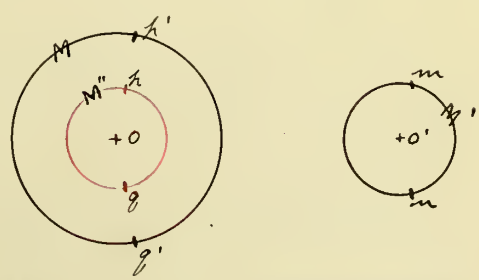
Find the intersection, c, of ab and p'q. (Art.,7)

Then pcq is the required path.

Proof: Any point in ab is equidistant from p and p'. (W.,Art.,160)

The straight line p'cq is shorter than any broken line p'dq. Therefore pcq is the required shortest path.

84. To draw a common tangent to two given circles.



Let M and M' be the given circles.

To draw a common tangent to M and M'.

Construct a circle M" with centre o and radius equal to the difference of the radii of M and M'.

From o', construct o'p and o'q tangent to M". (Art.,36,II)

Find the intersections p' and q' of op and oq with M. (Art.,4,I)

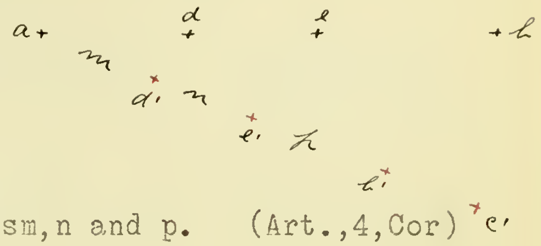
Construct o'm perpendicular to o'p and o'n perpendicular to o'q and find their intersections m and n with M'. (Arts.,12;4)

Then mp' and nq' are the required tangents.

Proof: pp' is parallel and equal to c'm. (W.,Art.,104)
Hence mp' is parallel to o'p (W.,Art.,183)
and must therefore be perpendicular to op' and o'm. (W.,Art.,107)
Therefore mp' is tangent to both circles M and M'.
Similarly nq' is a common tangent.

*Cor. Interior tangents may be constructed by using a circle with radius equal to the sum of the radii of the two given circles.

85. To divide a given straight line into parts proportional to any number of given lines.



Let ab be the given line and m,n and p the given parts.

To divide ab into parts proportional to m,n and p.

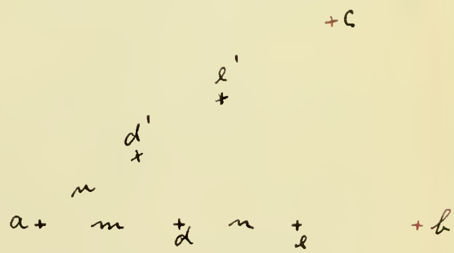
On any line ac, lay off the given parts m,n and p. (Art.,4,Cor)
Construct e'e and d'd parallel to b'b
and find their points of intersection d and e with ab. (Arts.,20;7)
Then ab is divided into the required parts by d and e.

Proof: ad:m=de:n=eb:p (W.,Art.,344)
Therefore these are the required parts.

86. To find a fourth proportional to three given straight lines.

This problem has been solved. (Art.,6)

87. To find a third proportional to two given straight lines.



Let m and n be the given straight lines.

To construct a third proprotional to m and n.

Take any angle as bac and lay off ad=m and de=ad'=n. (Art.,4,Cor.)
Construct ee' parallel to dd' (Art.,20)
and find the intersection e' of ee' and ac. (Art.,7)
Then d'e' is the required third proprotional.

Proof: ad:de=ad':d'e'. (W.,Art.,344)
Hence m:n=n:d'e'.
Therefore d'e' is the required third proprotional.

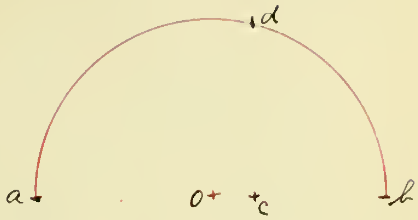
88. Construct x, if $x=ab/c$, (2) $x=a^2/c$.

These are special cases of Art.,6.

89. To find the mean proportional between two given straight lines.

Let ac and bc be the given straight lines.

To find a mean proprotional between ac and bc.



Find the sum ab of ac and bc. (Art.,5)
Bisect ab at o. (Art.,14)
Construct a semi-circle with oa as a radius.
Erect a perpendicular cd to ab (Art.,12,I)
and find its intersection d with the circumference. (Art.,4,II)
Then cd is the mean proprotional between ac and bc.

Proof: ac:cd=cd:ec. (W.,Art.,370)

Therefore cd is the required mean proprotional.

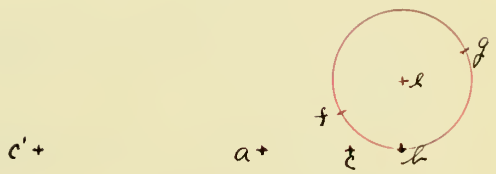
90. Construct x, if $x=\sqrt{ab}$.

If in the above construction we make a=ac and b=bc, then cd=x and the construction is completed.

91. To divide a given line in extreme and mean ratio.

Let ab be the given line.

To divide ab into extreme and mean ratio.



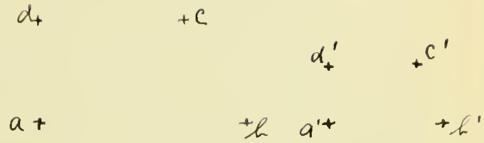
Construct be perpendicular to ab
and lay it off equal to 1/2 ab. (Art.,12;4,Cor)
With e as a centre and eb as a radius describe a circle and find its intersections f and g with ae. (Art.,4,I)
With a as a centre and af and ag respectively as radii, describe arcs and find their intersections c and c' with ab. (Art.,4,I)
Then ab is divided internally at c and externally at c' in extreme and mean ratio.

Proof: ag:ab=ab:af (W.,Art.,381)

$\overline{ab}^2=af\times ag=ac(af+fg)$	$\overline{ab}^2=ag\times af=c'a(ag-fg)$
$=ac(ac+ab)=\overline{ac}^2+ab\times ac.$	$=c'a(c'a-ab)=\overline{c'a}^2-ab\times c'a.$
Therefore $\overline{ab}^2-ab\times ac=\overline{ac}^2.$	Therefore $\overline{ab}^2+ab\times c'a=\overline{c'a}^2.$
Therefore $ab(ab-ac)=\overline{ac}^2.$	Therefore $ab(ab+c'a)=\overline{c'a}^2.$
Therefore $ab\times bc=\overline{ac}^2.$	Therefore $ab\times c'b=\overline{c'a}^2.$

Q.E.F.

92. Upon a given line homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.



Let abcd be the given polygon and a'b' the given line.

To construct a polygon on a'b' similar to abcd.

Construct ∠c'a'b'=∠cab and ∠c'b'a'=∠cba. (Art.,19)

Find the intersection c' of a'c' and b'c'. (Art.,7)

Construct ∠c'a'd'=∠cad and ∠a'c'd'=∠acd. (Art.,19)

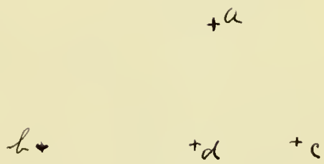
Find the intersection, d', of c'd' and a'd'. (Art.,7)

Then a'b'c'd' is the required polygon.

Proof: Triangle abc is similar to triangle a'b'c' and triangle acd is similar to triangle a'c'd'. (W.,Art.,355)

Therefore abcd is similar to a'b'c'd'. (W.,Art.,365)

93. To divide one side of a given triangle into segments proportional to the adjacent sides.



Let abc be the given triangle and bc the given side.

To divide bc in to segments proprotional to ab and ac.

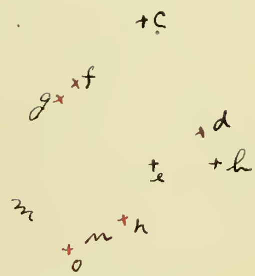
Construct the bisector ad of the angle bac. (Art.,15)

Find the intersection d of bc and ad. (Art.,7)

Then bc is divided at d into the required segments.

Proof: (W.,Art.,348)

94. To find in one side of a given triangle a point whose distance from the other sides shall be to each other in the given ratio m:n.



Let ab be the given side of the triangle abc, and m:n the given ratio.

To find a point e in ab so that ef/ed=m/n.

Construct the perpendicular ao to ac and lay it off equal to m. (Arts.,12;4,Cor)

Also construct oh perpendicular to bc and lay it off equal to n. Construct hg parallel to bc and ec parallel to og and find their intersections g and e with ac and ab. (Art.,20;7)

Then e is the required point.

Proof: Construct ed perpendicular to bc and ef perpendicular to ac. (Art.,12,II)

Find their intersections d and f with bc and ac. (Art.,7)

Then triangle ced is similar to triangle goh and triangle cef is similar to triangle goa. (W.,Art.,359)

Hence ce/ed=go/oh and ce/ef=go/ao. Dividing one of these equations by the other ,we get ef/ed=ao/oh=m/n.

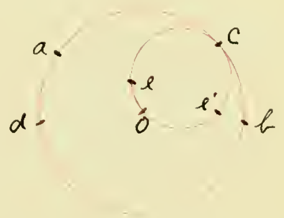
Therefore e is the required point.

95. Given an obtuse triangle; to draw a line from the vertex of th obtuse angle to the opposite side which shall be the mean proportional between the segments of that side.

Let abc be the given triangle.

Construct a line ec so that ae:ec=ec:eb.

Circumscribe a circle about abc. (Art.,34)



Construct a circle on oc as a diameter. (Art.,14)

Find its intersections e and e' with ab. (Art.,4,II)

Then ec or e'c is the required line.

Proof: Find the intersection d of ce with the circle. (Art.,4,II)

oe is perpendicular to dc. (W.,Art.,290)

Therefore de=ec (W.,Art.,245)

Then ae×eb=de×ec=ec (W.,Art.,378)

or ae:ec=ec:eb.

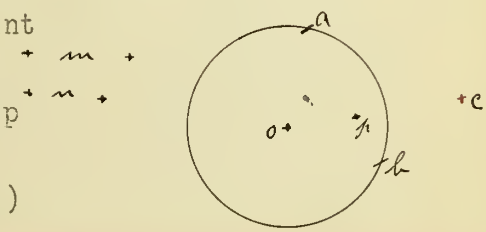
Similarly we get ae':e'c=e'c:e'b.

Therfore ec and e'c are the required lines.

96. Through a given point p within a given circle to draw a chord ab so that the ratio ap:bp shall equal the given ratio m:n.

Let o be the given circle, p the given point and m:n the given ratio.

To draw a chord ab through p so that ap:bp is equal to m:n.

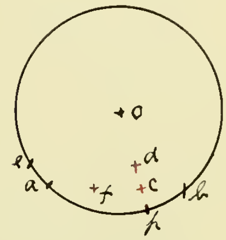


Extend op to c (Art.,4,Cor.)

making op:pc=m:n (Art.,6)
Lay off ac so that ac:ob=n:m. Extend ap to b. (Art.,4,Cor)
Then ab is the required chord.

Proof: From the above equations we have op:pc=ac:ob.
Also ∠apc=∠opb. (W.,Art.,93)
Then triangle apc is similar to triangle opb. (W.,Art.,357)
Hence ap:bp=pc:op=m:n.

97. To draw through a given point p in the arc subtended by a chord ab a chord which shall be bisected by ab.



Let ab be the given chord and p the given point.

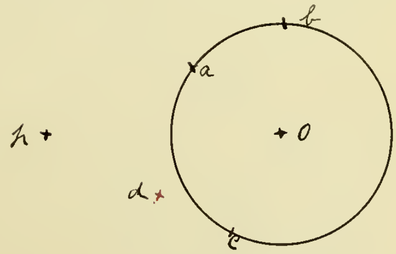
To draw a chord through p which shall be bisected by ab.

Find the intersection,c,of op and ab. (Art.,7)

Lay off cd=cp. (Art.,4,Cor)
Construct de parallel to ab. (Art.,20)
Find where it intersects the circumference. (Art.,4,II)
Find the intersection,f,of ep and ab. (Art.,7)
Then ep is the required chord.

Proof: In the triangle edp the line cf is constructed from the middle point c of the side dp,parallel to ed. It must,therefore,pass through the middle point f of ep. (W.,Art.,188)
Therefore ep is the required chord.

98. To draw through a given external point p a secant pab to a given circle so that the ratio pa:ab shall equal the given ratio m:n.



Let o be the given circle and p the given external point.

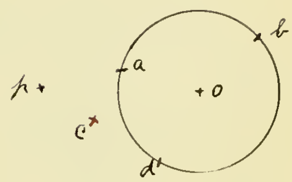
To construct a secant pab so that pa:ab=m:n.

Construct the tangent pc to the circle and divide it at d in the ratio m:n. (Arts.,36;85)
Construct pa so that pd:pa=pa:pc. (Art.,89)
Find the intersection,b,of ap with the circle. (Art.,4,II)
Then pab is the required secant.

Proof: We have pb:pc=pc:pa. (W., Art., 381)
From this equation and the one above we get
pd:pa=pc:pb.
Hence triangles pda and pcb are similar. (W., Art., 357)
Therefore pa:ab=pd:dc=m:n.
Therefore pb is the required secant.

99. To draw through a given external point p a secant pab to a given circle so that $\overline{ab}^2$=pa×pb.

Let o be the given circle and p the given external point.



Construct a secant pab such that ab =pa×pb.

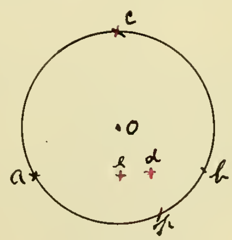
Construct the tangent pd to the circle and divide it into extreme and mean ratio at c. (Arts., 26; 91)

With p as a centre and pa as a radius locate the point a and find the intersection, b, of pa with the circle. (Art., 4, II)
Then pab is the required secant.

Proof: (1) pc:cd=cd:pd. Const.
(2) pb:pd=pd:pa=pd:cd. (W., Art., 381)
Combining (1) and (2) we have pc:cd or pc:pa=pd:pb
Hence the triangles pac and pbd are similar. (W., Art., 357)
Therefore (3) cd:pc=ab:pa.
From (1) pd=$\overline{cd}^2$/pc and substituting from (3) pd=ab/pa×pa=ab
From (2) $\overline{pd}^2$=pb×pa
Therefore $\overline{ab}^2$=pb×pa.
Therefore pb is the required secant.

100. To find a point p in the arc subtended by a given chord ab so that the ratio shall equal the given ratio m:n.

Let o be the given circle and ab the given chord.



To find a point p in the arc ab such that ap:bp=m:n.

Divide ab in the ratio m:n (Art., 85) and let d be the point of division.
Bisect ab at e, (Art., 14)
Erect a perpendicular ec to ab
and find its intersection c with the circle. (Arts., 12; 4, I)
Find the point p where cd cuts the circle. (Art., 4, II)

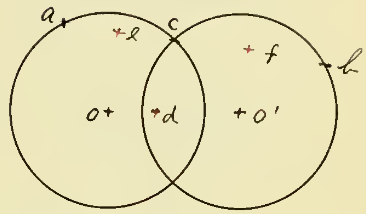
Then p is the required point.

Proof: Since c is the middle point of the arc acb then cp bisects the angle apb.

Therefore ap:pb=ad:bd=m:n. (W., Art., 348)

Therefore p is the required point.

101. To draw through one of the points of intersection of two circles a secant so that the two chords that are formed shall be in the given ratio m:n



Let o and o' be the given circles and c their point of intersection.

To construct a chord ab such that ac:cb=m:n.

Divide oo' in the ratio m:n (Art., 85)

Let d be the point of division.

Construct acb perpendicular to cd and find their intersection, a and b, with the circles. (Arts., 12; 4.II)

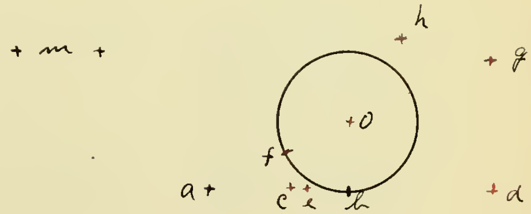
Then ab is the required chord.

Proof: Construct oe and o'f perpendicular to ab. (Art., 12, II)

Then ec:df=ac:cb=od:o'd=m:n. (W., Art., 344)

Therefore ab is the required chord.

102. Having given the greater segment of line diveded in extreme and mean ratio to construct the line.



Let m be the greater segment.

To construct the line of which m is the greater segment when it is divided in extreme and mean ratio.

Take any line ad and construct dg perpendicular to it. (Art., 12, I)

Lay off dg=1/2 ab. (Art., 4, Cor.)

With a as a centre and m as a radius, describe an arc cutting ag at f and ad at e. (Art., 4, I)

Construct fe perpendicular to ag. (Art., 12, I)

Find the intersection e of fe and ad. (Art., 7)

Construct the bisector, eh, of the angle fed. (Art., 15)

Find the intersection o of eh and ag. (Art., 7)

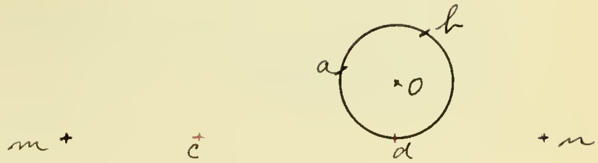
Construct ob perpendicular to ad. Then ab is the required line.

Proof: The construction is simply the converse of (Art.,91) which is to divide a line in extreme and mean ratio.

103. To construct a circle which shall pass through two given points and touch a given straight line.

Let a and b be the given points and mn the given straight line.

To draw a circle through a and b and tangent to mn.



Find the intersection, c, of ba and mn. (Art.,7)

Construct cd such that ca:cd=cd:cb. (Art.,89)

Draw a circle through a, b and d. (Art.,84)

Then this is the required circle.

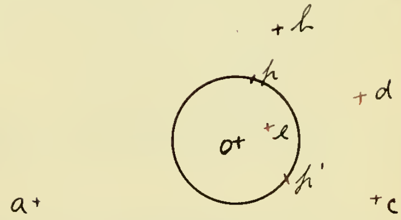
Proof: It passes through the two points a and b. Const.

Since ca:cd=cd:cb

then cd is tangent to the circle. (W.,Art.,381)

Therefore it is the required circle.

104. To construct a circle which shall pass through a given point and touch two given straight lines.



Let ab and ac be the two given lines and p the given point.

To construct a circle passing through p and tangent to both ab and ac.

Construct the bisector ad of ∠bac. (Art.,15)

Construct pp' perpendicular to ad, (Art.,12,II)

Find the intersection e of pp' and ad. (Art.,7)

Lay off ep' =ep. (Art.,4,Cor.)

Pass a circle through p and p' and tangent to ac. (Art.,103)

Then this is the required circle.

Proof: Since the circle passes through p and p' its centre must lie on ad. (W.,Art.,248)

Since the centre lies on ad and the circle touches ac, it must also touch ab. (W.,Art.,162)

Therefore o is the required circle.

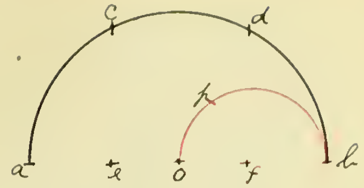
105. To inscribe a square in a semi circle.

Let o be the centre of the given semi-circle, acdb.

To inscribe a square in acdb.

Construct a semi-circle with ob as a diameter. (Art.,14)

Locate p such that op:pb=1:2. (Art.,100)



Lay off oe=of=op and fd=ec=bp. (Art.,4,Cor.)

Then ecdf is the required square.

Proof: Triangle ofd=triangle oec=triangle opb. (W.,Art.,150)

Then ∠oec=∠ofd=∠opd= a right angle. (W.,Art.,290)

Also of+oe=ef=fd=ec=cd.

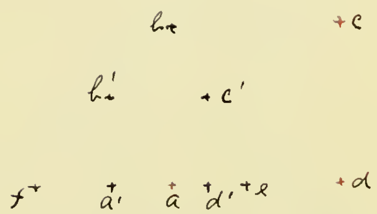
Therefore ecdf is the required square.

106. To inscribe a square in a given triangle.

Let fbe be the given triangle.

To inscribe a square in fbe.

Let a'b'c'd' be the required square.



Construct bc parallel to ef,(Art.,20)

Produce fc'

and find the intersection,c,of fc' and bc. (Arts.,4,Cor;7)

Construct ba and cd perpendicular to fe. (Art.,12,II)

Find the intersections,a and d,with ef. (Art.,7)

Then tri.fb'c' is similar to tri.fbc and tri.fc'd' is similar to tri.fcd. (W.,Art.,354)

Hence fc':fc=c'd':cd=c'b':cb. (W.,Art.,351)

Since c'b'=c'd',then cd=cb and since d' is a right angle,then d is a right angle and therefore abcd is a square.

Therefore to find the point c,construct the square abcd and find the intersection of fc and be. (Art.,7)

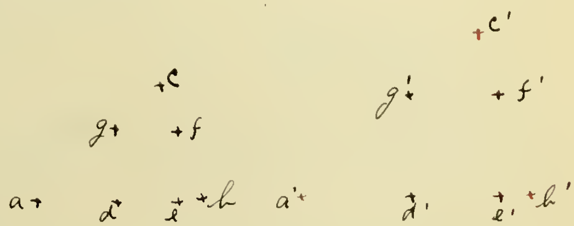
Having found the point c' we can easily construct the required square a'b'c'd' by constructing perpendicular and parallel lines.

107. To inscribe in a given triangle a rectangle similar to a given rectangle.

Let abc be the given triangle and d'e'f'g' the given rectangle.

To inscribe in abc a rectangle similar to d'e'f'g'.

Place the triangle abc so that ab will be parallel to d'e'. Through g' and f' construct c'a' parallel to ca

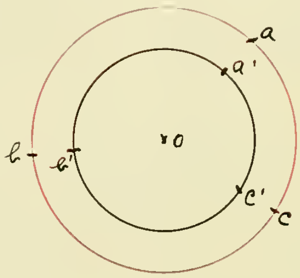


and c'b' parallel to cb. (Art.,20)
Extend d'e' to a' and b'. (Art.,4,Cor.)
Then triangles a'b'c' and abc are similar. (W.,Art.,354)
Locate g so that ag:gc=a'g':g'c'. (Art.,85)
Construct gf parallel to ab and then complete the rectangle defg. Then this is the required rectangle.

Proof: tri.cgf is similar to tri.c'g'f' and tri.agd to tri.a'g'd'. (W.,Art.,354)

Hence fg:f'g'=gd:g'd'. (W.,Art.,351)
Therefore defg is the required rectangle.

108. To inscribe in a circle a triangle similar to a given triangle.



Let the smaller circle whose centre is o be the given circle and abc the given triangle.

To inscribe in o a triangle similar to abc.

Circumscribe a circle around abc. (Art.,34)

Describe the given circle concentric to this circle.
Find the intersections a',b' and c' of oa,ob and oc with this circle. (Art.,4,I)

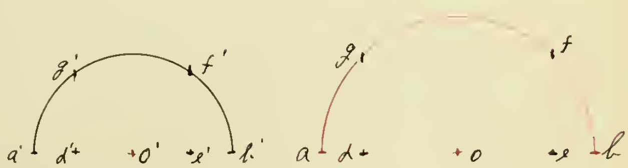
Then a'b'c' is the required triangle.

Proof: Triangles abc and a'b'c' are similar. (W.,Art.,354)
Therefore a'b'c' is the required triangle.

109. To inscribe in a given semicircle a rectangle similar to a given rectangle.

Let o' be the given semicircle and defg the given rectangle.

To inscribe in o' a rectangle similar to defg.

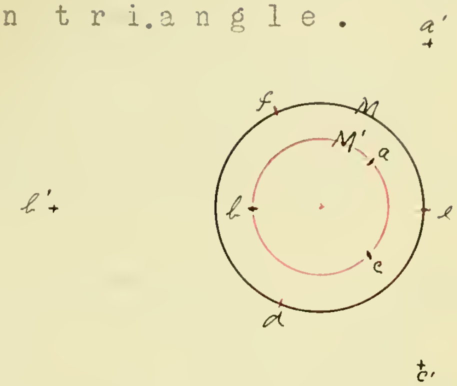


With o,the middle point of de,as a centre and of as a radius,describe a semicircle.
Construct o'e' such that o'e':o'f'=oe:of. (Art.,6)
Lay off o'd'=o'e'. (Art.,4,Cor)
Construct d'g' and e'f' perpendicular to a'b' and find their intersections g' and f' with the semicircle. (Arts.,12,4)
Then d'e'f'g' is the required rectangle.

Proof: The triangles o'e'f' and oef are similar. (W.,Art.,354)

Then o'e':oe=e'd':ed=e'f':ef etc. (W.,Art.,258)
Therefore d'e'f'g' is the required rectangle.

110. To circumscribe about a circle a triangle similar to a given triangle.



Let M be the given circle and abc the given triangle.

To circumscribe a triangle about M similar to abc.

Circumscribe a circle M' about the given triangle abc. (Art.,34)
Construct the given circle M concentric to M'.
Construct ae perpendicular to ac, of to ab and od to bc. (Art.,12,II)
Find their intersections e,f and d with the circle M. (Art.,4,I)
Construct tangents to M at e,f and d and find their intersections a',b' and c'. (Arts.,36;7)
Then a'b'c' is the required triangle.

Proof: Triangles abc and a'b'c' are similar. (W.,Art.,354)
Therefore a'b'c' is the required triangle.

111. To construct the expression x=2abc/de; that is 2ab/d×c/e.

This simply involves multiplication and division of lines.

See (Arts.,8,9)

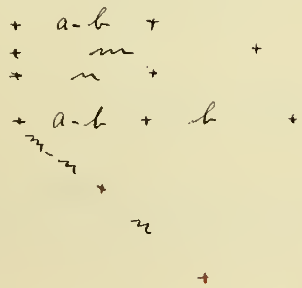
112. To construct two straight lines, having given their sum and their ratio.

Let ab be their sum and m:n their ratio.

To divide ab in the ratio m:n, see (Art.,85)

113. To construct two straight lines, having given their difference and their ratio.

Let a-b be their difference and m:n their ratio.



To construct a and b.

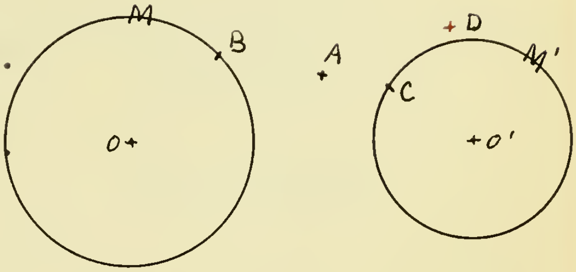
Construct m-n. (Art.,5)
Construct a fourth proportional to (m-n), n and (a-b). (Art.,6)
This fourth proportional is b, for

(m-n):n=(a-b):b. (W.,Art.,383)

Then construct (a-b)+b=a. (Art.,5)

Therefore we have constructed a and b .

114. Given two circles, with centres o and o', and a point A in their plane, to draw through the point A a straight line, meeting the circumference at A and B so that AB:AC=m:n.



Let M and M' be the given circles., A the given point and m:n the given ratio.

To construct a line through A meeting M and M' at B and C so that AB:AC=m:n.

Extend oA to D making m:n=oA:AD (Art.,4,Cor;6)

Construct DC such that m:n=oB:oC.(Art.,6)

With D as a centre and radius DC, locate the point C.

Find the intersection, B, of CA and M . (Art.,4,II)

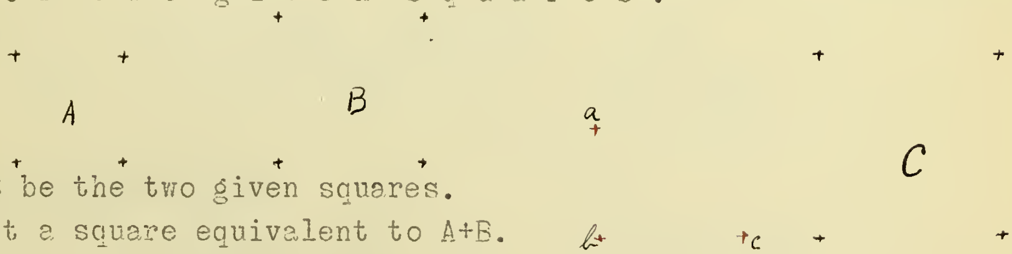
Then BC is the required line.

Proof: The triangles AoB and DAC are similar. (W.,Art.,345)

Then AB:AC=oA:AD=m:n.

Therefore BC is the required line.

115. To construct a square equivalent to the sum of two given squares.



Let A and B be the two given squares.

To construct a square equivalent to A+B.

Construct a right angle abc. (12,I)

Lay off ab equal to a side of A and bc equal to a side of B. (Art.,4,Cor.)

Construct a square, C, with ac as a side. (Arts.,12;4)

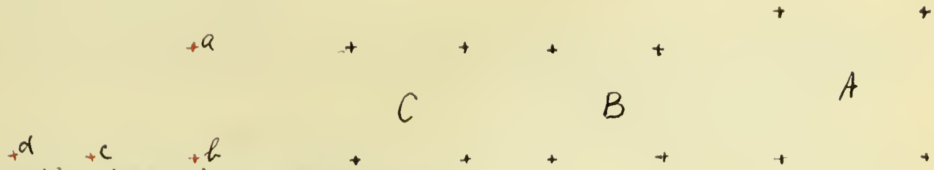
Then C is the required square.

Proof: $\overline{ac}^2=\overline{ab}^2+\overline{bc}^2$ (W.,Art.,415)

Therefore C is equivalent to A+B..

*Cor. By taking ac and the side of a third given square, we would be able to construct a square equivalent to the sum of three squares. This can be continued indefinitely .

116. To construct a square equivalent to thedifference of two given squares.



Let A and B be the two given squares.

To construct a square equivalent to A-B.

Construct a right angle abd. (Art.,12)

Lay off ab equal to a side of B, (Art.,4,Cor.)

With a as a centre and a radius equal to the side of A, strike an arc cutting bd at c. (Art.,4,II)

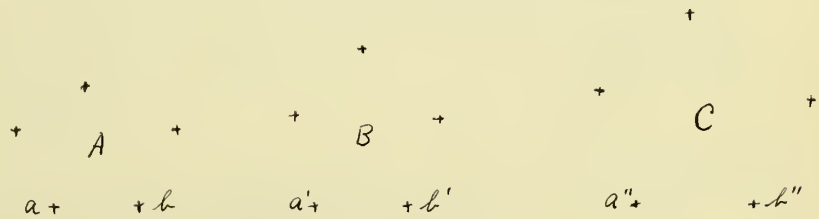
Construct a square, C, with side bc. (Arts.,12;4)

Then C is the required square.

Proof: $\overline{bc}^2=\overline{ac}^2-\overline{ab}^2$ (W.,Art.,416)

Therefore C is equivalent to A-B.

117. To construct a polygon similar to two given similar polygons and equivalent to their sum.



Let A and B be the two similar polygons and ab and a'b' the two homologous sides.

To construct a similar polygon equivalent to A+B.

Construct a"b" so that $\overline{a''b''}^2=\overline{a'b'}^2+\overline{ab}^2$. (Art.,115)

On a"b" construct a polygon C similar to B. (Art.,92)

Then C is the required polygon.

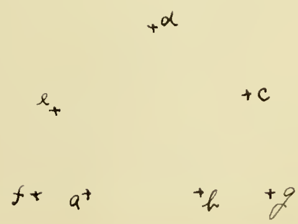
Proof: $\overline{a''b''}^2=\overline{a'b'}^2+\overline{ab}^2$ Const.

Also $A:C=\overline{ab}^2:\overline{a''b''}^2$ and $B:C=\overline{a'b'}^2:\overline{a''b''}^2$ (W.,Art.,412)

By addition $A+B:C=(\overline{ab}^2+\overline{a'b'}^2):\overline{a''b''}^2=1$.

Therefore C is equivalent to A+B.

118. To construct a triangle equivalent to a given polygon.



Let abcde be the given polygon.

To construct a triangle equivalent to abcde.

Construct cg parallel to bd

and ef parallel to ad, (Art.,20)
and find their intersections g and f with ab. (Art.,7)
Then dfg is the required triangle.

Proof: tri.dbg is equivalent to tri.dbc and tri.daf is equivalent to tri.dae. (W.,Art.,404)
In the polygon abcde and the tri.dfg, the part adb is common, tri.dbg is equivalent to tri.dbc and tri.daf to tri.dae.
Therefore tri.fdg is equivalent to abcde and is the required triangle.

*Cor. This same process may be applied to a polygon of any number of sides.

119. To construct a square equivalent to a given parallelogram.

Let abcd be the given parallelogram, b' its base and a' its altitude.

To construct a square equivalent to the parallelogram abcd.

Construct a mean proportional hp between a' and b'. (Art.,89)
Construct a square A having hp as a side. (Art.,12,4)
Then A is the required square.

Proof: hp =eh×gh=a'×b'. (W.,Art.,370)
Therefore A is equivalent to the parallelogram abcd.

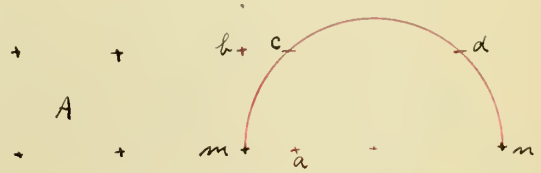
*Cor.I. A square may be constructed equivalent to a given triangle, by taking for its side the mean proportional between the base and half the altitude of the triangle.

*Cor.II. A square may be constructed equivalent to a given polygon, by first reducing the polygon to an equivalent triangle, and then constructing a square equivalent to the triangle.

120. To construct a parallelogram equivalent to a given square and having the sum of its base and altitude equal to a given line.

Let A be the given square and mn the sum of the base and altitude of the required parallelogram.

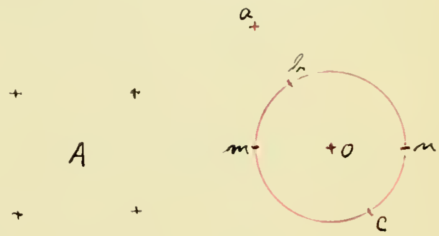
To construct a parallelogram equivalent to A and having the sum of its base and altitude equal to mn.



Construct a semicircle upon mn as a diameter. (Art.,14)
At m, erect a perpendicular mb to mn
and lay it off equal to the side of the given square A. (Art.,12,4)
Construct bd parallel to mn
and find its intersections c and d with the circle. (Arts.,20,4)
From c construct ca perpendicular to mn. (Art.,12,II)
Then any parallelogram constructed with an as a base and am as an altitude will satisfy the given conditions.

Proof: $\overline{ac}^2=\overline{bm}^2=A$.
Also $\overline{ac}^2=am\times an$. (W.,Art.,370)
Therefore A is equivalent to am×an.

121. To construct a parallelogram equivalent to a given square and having the difference of its base and altitude equal to a given line.



Let A be the given square and mn the difference between the base and altitude of the required parallelogram.

To construct a parallelogram equivalent to A and having the difference of its base and altitude equal to mn.

Construct a circle upon mn as a diameter. (Art.,14)
At m erect a perpendicular ma to mn
and make it equal to the side of the given square, A. (Arts.,12,4)
Find the intersections, b and c, of ao with the circle. (Art.,4,I)
Then a parallelogram constructed having ac as a base and ab as an altitude will satisfy the given conditions.

Proof: $\overline{am}^2=A$. Const.
$\overline{am}^2=ac\times ab$. (W.,Art.,381)
$ac-ab=bc=mn$.
Therefore A is equivalent to ac×ab.

122. To construct a polygon similar to a given polygon P and equivalent to a given polygon Q.

Let P and Q be the two given polygons, and ab a side of P.

To construct a polygon similar to P and equivalent

to Q.

Find squares equivalent to P and Q and let m and n respectively denote their sides. (Art., 119, Cor. II)

Construct $m:n=ab:a'b'$. (Art., 6)

Upon $a'b'$, homologous to ab, construct P' similar to P. (Art., 92)

Then P' is equivalent to Q and is the required polygon.

Proof: $m:n=ab:a'b'$. Const.

Hence $m^2:n^2=\overline{ab}^2:\overline{a'b'}^2$. (W., Art., 338)

But $P \simeq m^2$ and $Q \simeq n^2$. Const.

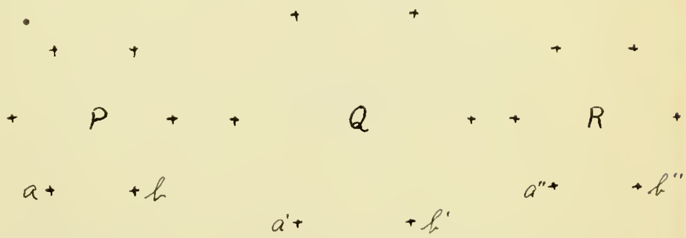
Hence $P:Q=m^2:n^2=\overline{ab}^2:\overline{a'b'}^2$.

But $P:P'=\overline{ab}^2:\overline{a'b'}^2$. (W., Art., 412)

Then $P:Q=P:P'$.

Therefore $P' \simeq Q$.

123. To construct a polygon similar to two given similar polygons and equivalent to their difference.



Let P and Q be the two given similar polygons.

To construct a polygon R similar to P and Q and equivalent to their difference.

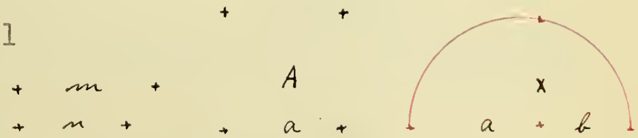
Construct $a''b''$ so that $\overline{a'b'}^2-\overline{ab}^2=\overline{a''b''}^2$. (Art., 116)

On $a''b''$ construct a polygon similar to P. (Art., 92)

Then this is the required polygon.

Proof: The proof is the same as for (Art., 117) after changing A+B to A-B.

124. To construct a square which shall have a given ratio to a given square.



Let A be the given square and n:m the given ratio.

To construct a square which shall be to A as n:m.

Construct a fourth proportional b to m, n and a. (Art., 6)

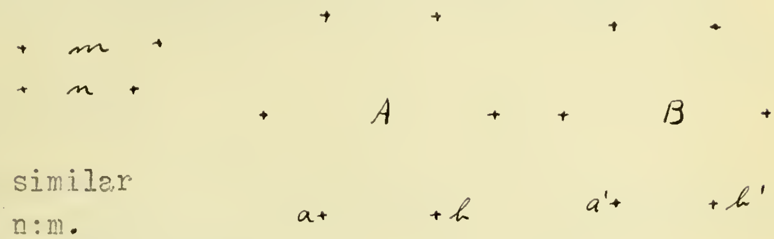
Construct a mean proportional x to a and b. (Art., 89)

Erect perpendiculars at the extremities of x and lay them off equal to x. (Arts., 12; 4, Cor.)

Then this is the required square.

Proof: a:x=x:b. (W., Art., 270)
Multiplying both sides of the equation by a/x we get $a^2:x^2=a:b$
But a:b=m:n.
Therefore $a^2:x^2=m:n$ or $x^2:a^2=n:m$.
Hence the square described on x will have the same ratio to A as n:m.

125. To construct a polygon similar to a given polygon and having a given ratio to it.



Let A be the given polygon and n:m the given ratio.

To construct a polygon similar to A, which shall be to A as n:m.

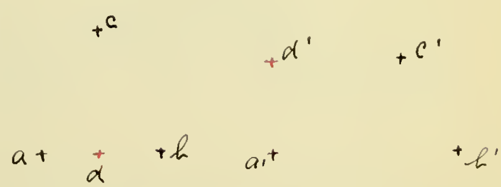
Construct a'b' such that $\overline{a'b'}^2:\overline{ab}^2=n:m$. (Art., 124)
Upon a'b', as a side homologous to ab, construct the polygon B similar to A. (Art., 92)
Then B is the required polygon.

Proof: $B:A=\overline{a'b'}^2:\overline{ab}^2$. (W., Art., 412)
But $\overline{a'b'}^2:\overline{ab}^2=n:m$. Const.
Therefore B:A=n:m.

126. To construct a triangle equivalent to a given triangle and having one side equal to a given length l.

Let abc be the given triangle and a'b' equal to the given side l.

To construct a triangle equivalent to abc and having a'b' for one side.



Construct cd perpendicular to ab. (Art., 12, II)
Find the intersection, d, of ab and cd. (Art., 7)
Construct a'd' such that a'b':ab=cd:a'd'. (Art., 6)
Then any triangle having a'b' for a base and a'd' for an altitude will satisfy the given conditions.

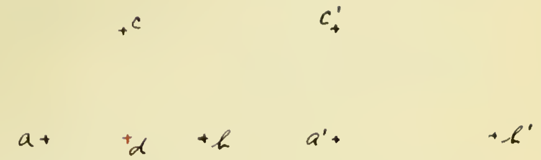
Proof: a'b':ab=cd:a'd'. Const.
Hence a'b'×a'd'=twice the area of triangle a'b'c'
and ab×cd=twice the area of triangle abc. (W., Art., 403)
Therefore abc is equivalent to a'b'c'.

129. To transform a given triangle into

an equivalent right triangle.

Let abc be the given triangle.

To construct a right triangle equivalent to abc.



Construct cd perpendicular to ab, (Art.,12,II)

Find its intersection, d, with ab. (Art.,7)

Construct a right triangle a'b'c', having ab and cd as legs. (Arts.,12; 4,Cor)

Then a'b'c' is the required triangle.

Proof: ab×cd=a'b'×c'a'. Const.

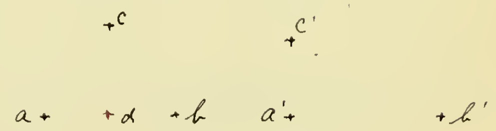
But ab×cd=twice the area of the triangle abc,

and a'b'×c'a'=twice the area of the triangle a'b'c'. (W.,Art.,403)

Therefore tri.abc is equivalent to tri.a'b'c'.

128. To transform a given triangle into an equivalent right triangle, having one leg equal to a given length.

Let abc be the given triangle and a'b' the given leg.



To transform abc into an equivalent right triangle having a'b' as one leg.

Construct a'c' such that a'b':ab=cd:a'c'. (Art.,6)

Then a right triangle constructed having a'b' and a'c' as legs will be the required triangle. (Arts.,12,4,Cor)

Proof: a'b'×a'c'=ab×cd. Const.

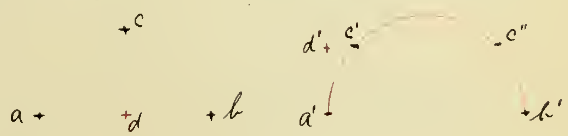
But a'b'×a'c'=twice the area of the triangle a'b'c'

and ab×cd=twice the area of the triangle abc. (W.,Art.,403)

Therefore tri.abc is equivalent to tri.a'b'c'.

129 To transform a given triangle into an equivalent right triangle, having the hypotenuse equal to a given length.

Let abc be the given triangle and a'b' the given hypotenuse.



To construct on the hypotenuse a'b' a right triangle equivalent to the triangle abc.

Construct cd perpendicular to ab and a'd' to a'b'. (Art.,12)

Construct a semicircle with a'b' as a diameter. (Art.,14)

Construct a'd' such that a'b':ab=cd:a'd'. (Art.,6)
Construct d'c"parallel to a'b'.
and intersecting the semicircle at c' and c". (Arts.,20;4,II)
Then a'b'c' or a'b'c" is the required triangle..

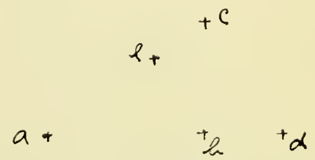
Proof: ∠a'c'b'=∠a'c"b'= a right angle. (W.,Art.,290)
ab×cd=a'b'×a'd'. Const.

But ab×cd= twice the area of the triangle abc
and a'b'×a'd'= twice the area of the triangle a'b'c'. (W.,Art.,403)
Therefore tri.abc is equivalent to tri.a'b'c'..

Note: The problem is impossible if a'd' greater than 1/2 a'b'.

130. To transform a triangle abc into an equivalent triangle, having a side equal to a given length l, and an angle equal to angle bac.

Let abc be the given triangle, ad=l the given side and bac the given angle.



To construct a triangle equivalent to abc, having ad as one side and bac as an angle.

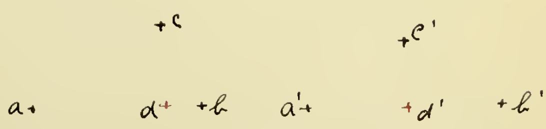
Construct be parallel to cd and find its intersection, e, with ac. (Arts.,29;7)
Then aed is the required triangle.

Proof: tri.bec ≎ tri.bed. (W.,Art.,404)
Therefore tri.aed ≎ tri.abc.

131. To transform a given triangle into an equivalent isosceles triangle, having the base equal to a given length.

Let abc be the given triangle and a'b' the given base.

To construct an isosceles triangle having a'b' as a base and equivalent to the triangle abc.



Construct c'd' perpendicular to a'b' at its middle point. (Art.,12,I)
Construct c'd' such that a'b':ab=cd:c'd'. (Art.,6)
Then a'b'c' is the required triangle.

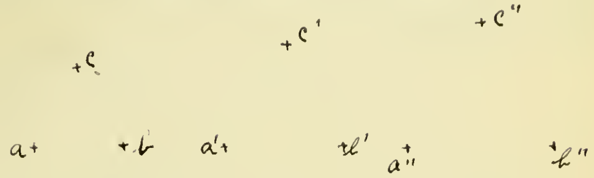
Proof: c'a'=c'b'. (W.,Art.,959)
Hence triangle a'b'c' is isosceles.
Also ab×cd= twice the area of the triangle abc
and a'b'×c'd'= twice the area of the triangle a'b'c'. (W.,Art.,403)

Therefore tri.abc is equivalent to tri.a'b'c'.

132. To construct a triangle equivalent to;

I. The sum of two given triangles.

Let abc and a'b'c' be the two given triangles.



To construct a triangle equivalent to the sum of abc and a'b'c'.

Construct squares A and B equivalent to the triangles abc and a'b'c'. (Art.,119,Cor.,I)

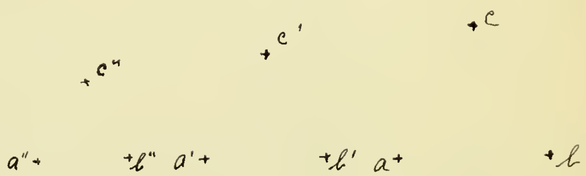
Construct a square C equivalent to A+B. (Art.,115)

Construct a triangle a"b"c" equivalent to C. (Art.,118)

Then a"b"c" is the required triangle.

II. The difference of two given triangles.

Let abc and a'b'c' be the two given triangles.



To construct a triangle equivalent to the difference of abc and a'b'c'.

Construct squares A and B equivalent to the triangles abc and a'b'c'.

Construct a square C equivwlent to A-B. (Art.,116)

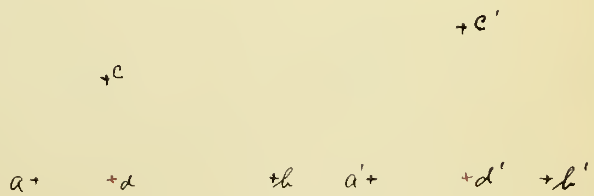
Construct a triangle a"b"c" equivalent to C. (Art.,118)

Then a"b"c" is the required triangle.

133. To transform a given triangle into an equivalent equilateral triangle.

Let abc be the given triangle.

To construct an equilateral triangle equivalent to abc.



Construct cd perpendicular to ab. (Art.,12,II)

Find the intersection,d,of ab and cd. (Art.,7)

Construct $2\times\overline{ab}\times\overline{cd}$, (Art.,8)

and then $\sqrt{2\times\overline{ab}\times\overline{cd}}$. (Art.,10)

Construct $\sqrt[4]{3}=\sqrt{\sqrt{3}}$, (Art.,10)

and then $(\sqrt{2\times\overline{ab}\times\overline{cd}})\div\sqrt[4]{3}$. (Art.,9)

Construct the equilateral triangle a'b'c', making $a'b'=(\sqrt{2\times\overline{ab}\times\overline{cd}})\div\sqrt[4]{3}$. (Art.,32)

Then a'b'c' is the required triangle.

Proof: $\overline{c'd'}^2=(\sqrt{(2\times\overline{ab}\times\overline{cd})}\div\sqrt[4]{3})^2-1/4(\sqrt{(2\times\overline{ab}\times\overline{cd})}\div\sqrt[4]{3})^2=1/2\overline{ab}\times\overline{cd}\sqrt{3}$. (W., Art. 371)

Hence $c'd'=(\sqrt{\overline{ab}\times\overline{cd}}\times\sqrt[4]{3})\div\sqrt{2}$.

Triangle $a'b'c' \Bumpeq 1/2(\sqrt{\overline{ab}\times\overline{cd}}\times\sqrt[4]{3})\div\sqrt{2})(\sqrt{2\times\overline{ab}\times\overline{cd}})\div\sqrt[4]{3})=1/2\overline{ab}\times\overline{cd}$. (W., Art., 403)

But triangle $abc=1/2\overline{ab}\times\overline{cd}$, (W., Art., 403)

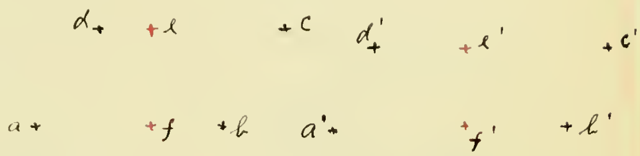
and therefore triangle abc $\Bumpeq$ triangle to a'b'c'.

Therefore a'b'c' is the required triangle.

134. To transform a parallelogram into an equivalent:

I. Parallelogram having one side equal to a given length.

Let abcd be the given parallelogram and a'b' the given side.



To construct a parallelogram with a side a'b' and equivalent to abcd.

Construct ef perpendicular to ab and e'f' to a'b'. (Art., 12)

Construct e'f' such that a'b':ab=ef:e'f'. (Art., 6)

Construct d'c' parallel to a'b' and at a distance e'f' from it. (Art., 20)

With a' and b' as centres and any radius describe arcs cutting b'c' in b' and c'. (Art., 4)

Then a'b'c'd' is the required parallelogr am.

Proof: $ab\times ef=a'b'\times e'f'$. Const.

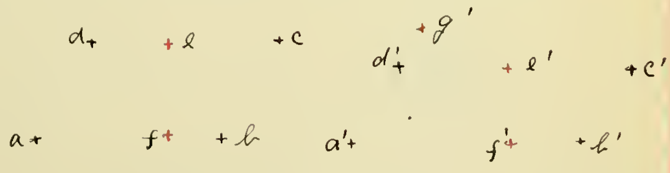
But $ab\times ef=$ area of the parallelogram abcd

and $a'b'\times e'f'=$ area of the parallelogram a'b'c'd'. (W., Art., 400)

Therefore, parallelogram abcd $\Bumpeq$ parallelogram a'b'c'd'.

II. Parallelogram having one angle equal to a given angle.

Let abcd be the given parallelogram and b'a'g' the given angle.



To construct a parallelogram equivalent to abcd and having b'a'g' as one angle.

Lay off a'b' any arbitary length. (Art., 4, Cor)

Construct ef perpendicular to ab and e'f' to a'b'. (Art., 12)

Lay off e'f' such that a'b':ab=ef:e'f'. (Art., 6)

Construct d'e'c' parallel and equal to a'b'. (Art., 20)

Then a'b'c'd' is the required parallelogram.

Proof: $\angle$d'a'b' is the given angle.

ab×ef=a'b'×e'f'. Const.

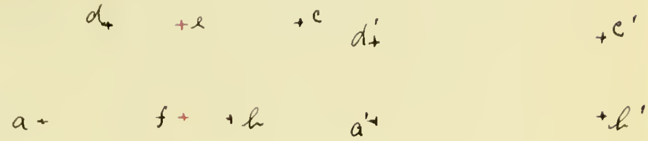
But ab×ef= area of the parallelogram abcd

and a'b'×e'f'= area of the parallelogram a'b'c'd'. (W.,Art.,400)

Therefore, parallelogram abcd $\approx$ parallelogram a'b'c'd'.

III. Rectangle having a given altitude.

Let abcd be the given parallelogram and a'b' the altitude of the required rectangle.



To construct a rectangle with altitude a'd' and equivalent to abcd.

Construct a'b' such that a'd':ef=ab:a'b'. (Art.,6)

Erect perpendiculars a'd' and b'c' to a'b' and lay them off equal to a'd'. (Arts.,12;4,Cor.)

Then a'b'c'd' is the required rectangle.

Proof: ab×ef=a'b'×a'd'. Const.

But ab×ef= area of the parallelogram abcd, (W.,Art.,400)

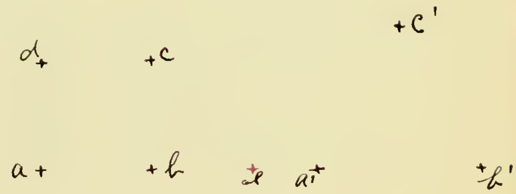
and a'b'×a'd'= area of the rectangle a'b'c'd'. (W.,Art.,398)

Therefore, parallelogram abcd $\approx$ rectangle a'b'c'd'.

135. To transform a square into an equivalent: .

I. Equilateral triangle.

Let abcd be the given square.



To construct an equilateral triangle equivalent to abcd.

Construct triangle ade $\approx$ square abcd. (Art.,118)

Construct the equilateral triangle a'b'c' $\approx$ triangle ade. (Art.,133)

Then a'b'c' is the required triangle.

Proof: Triangle ade $\approx$ square abcd. Const.

Triangle ade $\approx$ triangle a'b'c'. Const.

Therefore, triangle a'b'c' $\approx$ square abcd.

II. Right triangle having one leg equal to a given length.

Let abcd be the given square and a'b' the given leg of the required triangle.

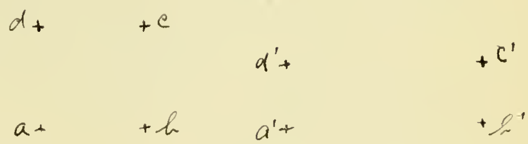
To construct a right triangle equivalent to abcd and having a'b' for one leg.

Construct b'd' such that a'b':ab=ab:b'd'. (Art.,6)
Construct b'c' perpendicular to a'b' and equal to 2b'd'. (Art.,12;8)
Then a'b'c' is the required triangle.

Proof: $\overline{ab}^2$=a'b'×b'd'. Const.
But $\overline{ab}^2$ = area of the square abcd. (W.,Art.,398)
And 1/2a'b'×c'b'=a'b'×b'd'=area of triangle a'b'c'.(W.,Art.,403)
Therefore triangle a'b'c' ≎ square abcd.

III. Rectangle having one side equal to a given length.

Let abcd be the given square and a'b' the side of the required rectangle.



To construct a rectangle equivalent to abcd and having a'b' as one side.

Construct c'b' such that a'b':ab=ab:c'b'. (Art.,87)
Construct a'd' and b'c' perpendicular to a'b' and lay them off equal to c'b'. (Arts.,12;4,Cor.)
Then a'b'c'd' is the required rectangle.

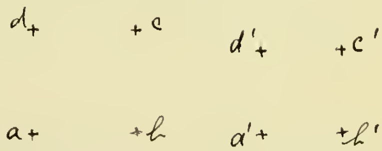
Proof: $\overline{ab}^2$=a'b'×c'b' Const.
But $\overline{ab}^2$= area of the square abcd. (W.,Art.,398)
And a'b'×c'b'= area of rectangle a'b'c'd' (W.,Art.,398)
Therefore,square abcd ≎ rectangle a'b'c'd'.

136. To construct a square equivalent to;

I. Five eighths of a given square.

Let abcd be the given square.

To construct a square equivalent to five eighths of abcd.



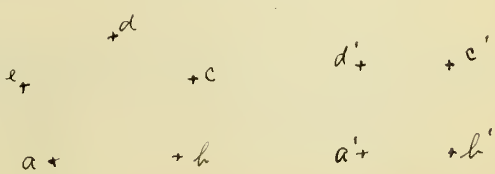
Construct a line equal to 5/8 ab. (Arts.,8,9)
Construct a'b' such that ab:a'b'=a'b':5/8 ab. (Art.,89)
Construct a'd' and b'c' perpendicular to a'b' and lay them off equal to a'b'. (Arts.,12;4,Cor.)
Then a'b'c'd' is the required square.

Proof: $\overline{a'b'}^2$ =5/8 $\overline{ab}^2$. Const.
Therefore area of square a'b'c'd' ≎ 5/8 area of square abcd.

II. Three fifths of a given pentagon.

Let abcd be the given pentagon.

To construct a square equivalent to three fifths of abcde.



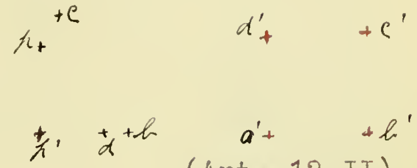
Construct triangleA ≎ pentagon

abcde. (Art.,118)
Construct square B $\approx$ triangle A. (Art.,119,Cor.I)
Construct square a'b'c'd' $\approx$ 3/5 square B. (Art.,136,I)
Then a'b'c'd' is the required square.

Proof: Pentagon abcde $\approx$ triangle A $\approx$ square B, Const.
and square a'b'c'd' $\approx$ 3/5 square B. Const.
Therefore square a'b'c'd' $\approx$ 3/5 pentagon abcde.

137. To divide a given triangle into two equivalent parts by a line through a given point p in one of the sides.

Let abc be the given triangle and p the given point in the side ac.



To draw a line through p dividing abc into two equivalent parts.

Construct pp' perpendicular to ab. (Art.,12,II)
Construct square a'b'c'd' $\approx$ triangle abc . (Art.,119,Cor.I)
Construct ad such that pp':a'b'=a'b':ad. (Art.,87)
Lay off ad on ab. (Art.,4,Cor.)
Then pd divides the triangle into the required parts.

Proof: $pp' \times ad = \overline{a'b'}^2$. Const.
Hence $pp' \times ad$ = area of triangle abc.
But 1/2 $pp' \times ad$ = area of triangle adp. (Art.,403)
Therefore triangle adp $\approx$ 1/2 triangle abc.
Then pdbc $\approx$ 1/2 triangle abc.

Note. If ad is greater than ab, use the perpendicular from p to bc instead of pp' and lay off ad on cb.

138. To find a point within a triangle, such that the lines joining this point to the vertices shall divide the triangle into three equivalent parts.

Let abc be the given triangle.

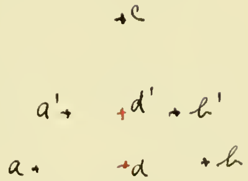
To find a point p such that triangle apb $\approx$ triangle apc $\approx$ triangle bpc.

Construct cd perpendicular to ab and be to ac. (Art.,12,II)
Construct 1/3 cd and 1/3 be. (Art.,23)
Construct fg parallel to ab and at a distance 1/3 cd from it and f'g' parallel to ac and at a distance 1/3 be from it. (Art.,20)
Find the intersection, p, of fg and f'g'. (Art.,7)

Then p is the required point.

Proof: Area of triangle apc=1/2 ac×cp=1/2 ac×1/3 eb. (W., Art., 403)
But 1/2 ac×1/3 eb 1/3 triangle abc.
Similarly triangle apb triangle bpc 1/3 triangle abc.
Therefore p is the required point.

139. To divide a given triangle into two equivalent parts by a line parallel to one of the sides.



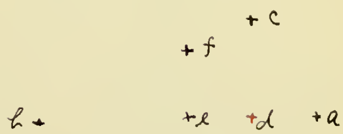
Let abc be the given triangle.

To divide abc into two equivalent parts by a line parallel to ab.

Construct cd perpendicular to ab and find its intersection, d, with ab. (Arts., 12; 7)
Construct cd'=cd/$\sqrt{2}$. (Arts., 9; 10)
Lay off cd' on cd. (Art., 4, Cor.)
Construct a'b' through d' parallel to ab and find its intersections with ac and bc. (Arts., 20-7)
Then a'b' is the required line.

Proof: Triangle abc and a'b'c' are similar. (Art., 259)
Therefore tri.abc:tri.a'b'c=$\overline{cd}^2:\overline{cd}^2/2$=2:1 (W., Art., 412)
Hence tri.a'b'c'$\approx$ 1/2 tri.abc.
Then abb'a'$\approx$ 1/2 tri.abc.
Therefore a'b' is the required line.

140. To divide a given triangle into two equivalent parts by a line perpendicular to one of the sides.



Let abc be the given triangle.

To draw a line perpendicular to ab dividing abc into two equivalent parts.

Construct cd and ef perpendicular to ab (Art., 12, II)
and suppose ef is the required line.
For brevity let be=y, ef=x, bd=m, ad=n, cd=p.
Then 1/2 p(n+m)=area of triangle abc. (W., Art., 403)
And (1) 1/2 xy=1/4 p(n+m). Const.
Also y:x=m:p. (W., Art., 342)
Therefore mx=py or x=py/m.
Substituting this in (1) and solving, we get
$y^2=(np+p^2)\div 2$ or $y=\sqrt{(np+p^2)\div 2}$ or be=$\sqrt{(ad\times cd+cd^2)\div 2}$.
Hence, construct be=$\sqrt{(\overline{ad}\times\overline{cd}+\overline{cd}^2)\div 2}$ (Arts., 7, 8, 9, 10)

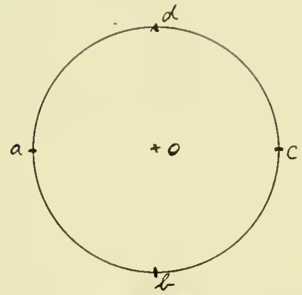
From e construct ef perpendicular to ab. (Art.,12,I)
Then ef is the required line.

Proof: From equation (1), $1/2\ ef \times be = 1/4\ cd \times ab$.
Therefore triangle bef $\backsimeq$ 1/2 triangle abc.
Therefore ef is the required line.

141. T o i n s c r i b e a s q u a r e i n a g i v e n c i r c l e.

Let o be the given circle.

To inscribe a square in the circle o.



Construct two diameters, ac and bd, perpendicular to each other. (Art.,12)
Find their intersections, a, b, c and d with the circle. (Art.,4,I)
Then abcd is the required square.

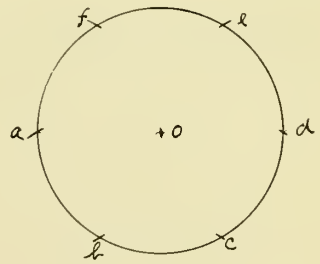
Proof: Angles abc, bcd, etc., are right angles. (W.,Art.,290)
Also ab=bc=cd=da. (W.,Art.,241)
Hence abcd is a square.

*Cor. By bisecting the arcs ab, bc, etc., and repeating the operation, we can form regular inscribed polygons of eight, sixteen, etc., sides.

142. T o i n s c r i b e a r e g u l a r h e x a g o n i n a g i v e n c i r c l e.

Let o be the given circle.

To inscribe a regular hexagon in o.



Begin with any point, as a, on the circumference and apply the radius, oa, six times as a chord, cutting the circumference at b, c, d, etc.

Then abcdef is the inscribed hexagon.

Proof: ab=bc=cd=etc. Const.
The triangle aob is equiangular. (W.,Art.,146)
Hence $\angle aob = 60°$ and arc $ab = 60^c = 1/6$ of a circumference.
Therefore ab is the side of a regular inscribed hexagon.

*Cor.I. ace is an inscribed equilateral triangle.

*Cor.II. By bisecting the arcs ab, bc, etc., and continuing the process we get regular inscribed polygons of twelve, twenty-four, etc., sides.

143. T o i n s c r i b e a r e g u l a r d e c a g o n i n a g i v e n c i r c l e.

Let o be the given circle.

To inscribe a regular decagon in o.

Divide the radius oa into extreme and mean ratio such that oa:oc=oc:ac. (Art.,91)

Beginning with a as a centre and using oc as a radius,describe arcs locating the points b,d,etc.

Then abd....is the required decagon.

Proof: oa:oc=oc:ac and oc=ab. Const.

Hence oa:ab=ab:oc.

Also $\angle a$ is common to the two triangles oab and cab.

Hence the triangles oab and cab are similar. (W.,Art.,357)

Since tri.oab is isosceles,then tri.abc is isosceles and ab=bc=oc.

Then tri.bco is isosceles and $\angle o=\angle cbo$.

But,exterior $\angle acb=\angle o+\angle cbo=2\angle c$. (W.,Art.,137)

Then $\angle oab=\angle oba=2\angle o$.

Hence the sum of the angles of the tri.abo=$5\angle c$=2 right angles.

Then $\angle o$=1/5 of 2 right angles=1/10 of 4 right angles.

Therefore,arc ab is 1/10 of the circumference and chord ab is the side of a regular inscribed decagon.

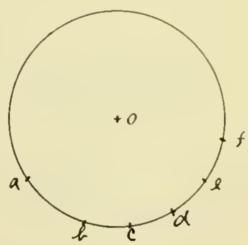
*Cor.I. By joining the alternate vertices of a regular inscribed decagon,a regular pentagon is formed.

*Cor.II. By bisecting the arcs ab,bd,etc.,and continuing the process we get regular inscribed polygons of twenty,forty,etc.,sides.

144. To inscribe in a given circle a regular pentadecagon,or polygon of fifteen sides.

Let o be the given circle.

To inscribe in o a regular polygon of fifteen sides.



Lay off the chord ac equal to the radius of the circle and ab equal to the side of a regular inscribed decagon.

Then bc is the side of the required polygon and applying it fifteen times as a chord we get the required polygon.

Proof: The arc ac is 1/6 of the circumference. (Art.,142)

The arc ab is 1/10 of the circumference. Const.

Hence,the arc bc is 1/6-1/10 or 1/15 of the circumference.

Then bc is the side of a regular inscribed pentadecagon.

*Cor. By bisecting the arcs bc,cd,etc.,we get regular polygons of thirty,sixty,etc.,sides.

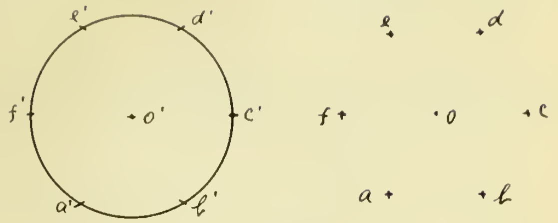
145. To inscribe in a given circle a reg-

u l a r p o l y g o n s i m i l a r t o a g i v e n r e g u l a r p o l y g o n .

Let c' be the given circle and abcdef the given regular polygon.

To inscribe in o' a regular polygon similar to abcdef.

Let o be the centre of the given polygon.



At o' construct ∠e'o'd'=∠eod. (Art.,19)

Find the intersections,e' and d', of oe' and od' with the circle. (Art.,4,I)

Then e'd' is the side of the required polygon and applying it six times as a chord we get the required polygon.

Proof: Each polygon has as many sides as the ∠c, or ∠o', is contained times in four right angles. Therefore, the polygon a'b'c' is similar to the polygon abc....etc. (W.,Art.,445)

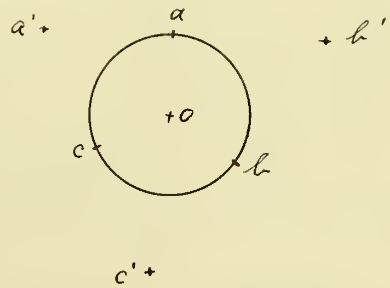
146. T o c i r c u m s c r i b e a n e q u i l a t e r a l t r i- a n g l e a b o u t a g i v e n c i r c l e .

Let o be the given circle.

To circumscribe an equilateral triangle about c.

Divide the given circumference into three equal parts by a,b and c. (Art.,142,Cor.I)

Construct tangents to the circle at a,b and c,



and find their intersections a'b'c'. (Arts.,36,7)

Then a'b'c' is the required triangle.

Proof: abc is a regular inscribed triangle. Const.

Therefore a'b'c' is a regular circumscribed triangle. (W.,Art.,449)

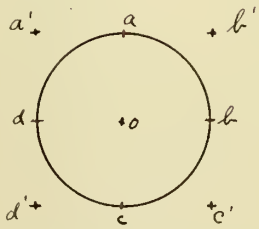
147. T o c i r c u m s c r i b e a s q u a r e a b o u t a g i v- e n c i r c l e .

Let o be the given circle.

To circumscribe a square about o.

Inscribe a square,abcd,in the given circle. (Art.,141)

Construct tangents to the circle at the points a,b,c,d, and find their intersections a'b'c' and d'.(Arts.,36;7)

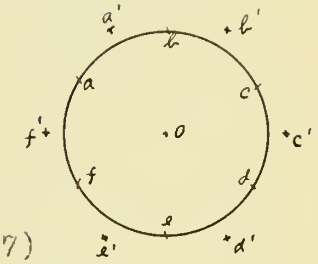


Then a'b'c'd' is the required square.

Proof: abcd is an inscribed square. Const.

Then a'b'c'd' is a circumscribrd square. (W.,Art.,440)

148. To circumscribe a regular hexagon about a given circle.



Let o be the given circle.

To circumscribe a regular hexagon about c.

Inscribe a regular hexagon in o.(Art.,142)

Construct tangents tc o at the points a,b,c etc.

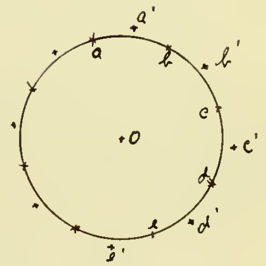
and find their intersections a',b'c' etc. (Arts.,36;7)

Then a'b'c'd'e'f' is the required hexagon.

Proof: abcdef is a regular inscribed hexagon. Const.

Then a'b'c'd'e'f' is a regular circumscribed hexagon. (W.,Art.,440)

149. To circumscribe a regular octagon about a given circle.



Let o be the given circle.

To circumscribe a regular octagon about o.

Inscribe a regular octagon in c.

(Art.,141,Cor.)

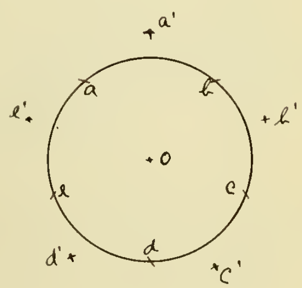
Construct tangents to the circle at the vertices a,b,c,etc.,of the regular inscribed octagon, and find their intersections a',b',c',etc. (Arts.,36;7)

Then a'b'c',etc.,is the required octagon.

Proof: abcd..is a regular inscribed octagon. Const.

Then a'b'c'd'....is a regular circumscribed octagon. (W.,Art.,440)

150. To circumscribe a regular pentagon about a given circle.



Let o be the given circle.

To circumscribe a regular pentagon about o.

Inscribe a regular pentagon in o.

(Art.,143,Cor.I)

Construct tangents to the circle at the vertices a,b,c,et.,of the regular inscribed pentagon,

and find the points of intersection,a',b',c' etc.,of these tangents.

Arts.,36,7)

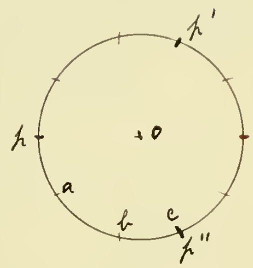
Then a'b'c'd'e' is the required pentagon.

Proof: abcde is a regular inscribed pentagon. Const.

Then a'b'c'd'e' is a regular circumscribed pentagon. (W., Art., 440)

151. To draw through a given point a line so as to divide a given circumference into two parts having the ratio 3:7.

Let o be the given circle and p the given point.



To draw through p a line dividing the circumference of o into two parts having the ratio 3:7.

Inscribe in o a regular decagon using p as one of the vertices.

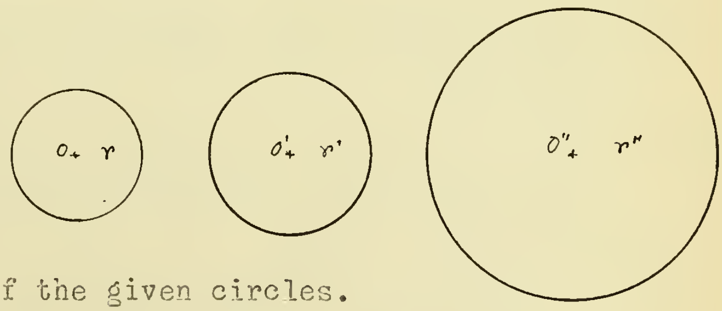
Then pp' or pp" cutting off three of the equal arcs is the required line.

Proof: Since it is a regular polygon the subtended arcs are equal. (W., Art., 241)

Then a line cutting off three of them would divide the circumference in the ratio 3:7.

152. To construct a circumference equal to the sum of two given circumferences.

Let o and o' be the two given circles.



To construct a circle whose circumference will be the sum of the circumferences of o and o'.

Let r and r' be the radii of the given circles.

Construct a circle o" with $r''=r+r'$ (Art., 5)

Then o" is the required circle.

Proof: The circumference of the circles are respectively $2\pi r$, $2\pi r'$ and $2\pi r''$. (W., Art., 458)

We must have, therefore, $2\pi r''=2\pi r+2\pi r'$ or $r''=r+r'$. Hence if we construct a circle with radius $r''=r+r'$ it will be the required circle.

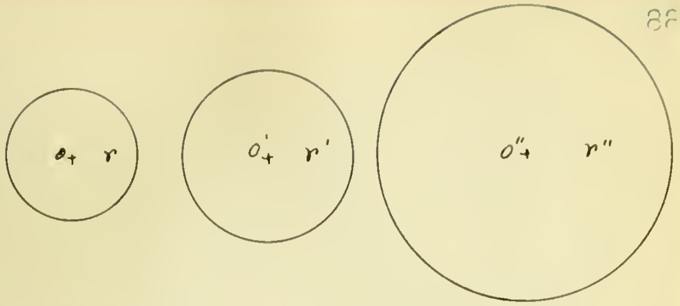
153. To construct a circumference equal to the difference of two given circumferences.

Let o' and o" be the two given circles.

To construct a circle whose circumference will be the difference of

the circumferences of o" and o'.

Let r' and r" be the radii of the given circles.



Construct a circle o with $r=r''-r'$. (Art.,5)

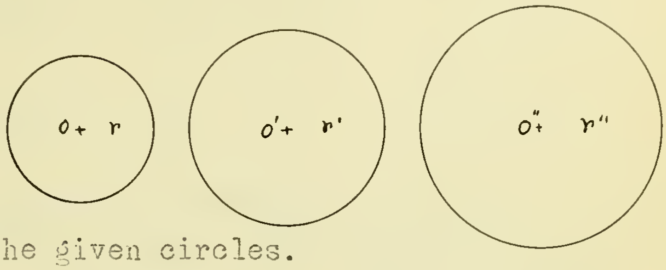
Then o is the required circle.

Proof: The circumference of the circles are respectively $2\pi r, 2\pi r', 2\pi r''$. (W., Art., 458)

We must, therefore, have $2\pi r=2\pi r''-2\pi r'$ or $r=r''-r'$. Hence if we construct a circle with radius $r=r''-r'$ it will be the required circle.

154. To construct a circle equivalent to the sum of two given circles.

Let o and o' be the two given circles.



To construct a circle equivalent to the sum of the two given circles.

Let r and r' be the radii of the given circles.

Construct $r''=\sqrt{r^2+r'^2}$. (Art., 13, Cor.)

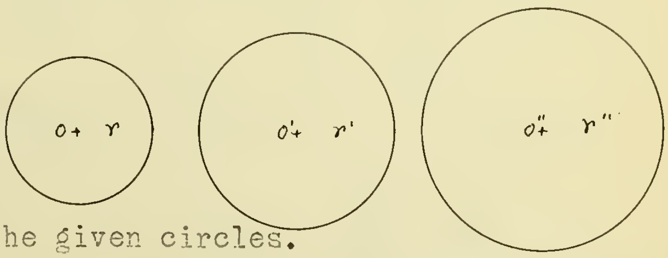
Then construct a circle o" with r" as a radius. Then o" is the required circle.

Proof: Squaring $r''=\sqrt{r^2+r'^2}$ and multiplying by π we get

$\pi r''^2=\pi r^2+\pi r'^2$. Hence o" ⋍ o+o'. (W., Art., 463)

155. To construct a circle equivalent to the difference of two given circles.

Let o' and o" be the two given circles.



To construct a circle equivalent to the difference of the two given circles.

Let r' and r" be the radii of the given circles.

Construct $r=\sqrt{r''^2-r'^2}$ (Art., 13, Cor.)

Then construct a circle o with the radius r. Then o is the required circle.

Proof: squaring $r=\sqrt{r''^2-r'^2}$ and multiplying by π we get

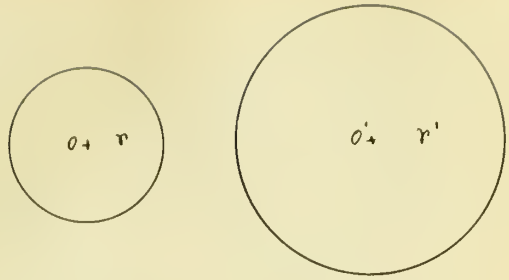
$\pi r^2=\pi r''^2-\pi r'^2$. Hence o ⋍ o"-o'. (W., Art., 463)

156. To construct a circle equivalent to three times a given circle.

Let o be the given circle.

To construct a circle equivalent to three times o.

Let r be the radius of the given circle.



Construct $r'=r\sqrt{3}$. (Arts.,10;8)
Construct a circle o' with radius r'.
Then o' is the required circle.

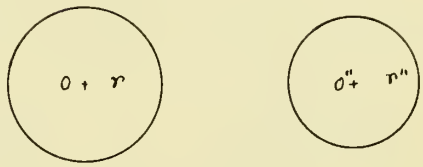
Proof: Squaring $r'=r\sqrt{3}$ and multiplying by π we get $\pi r'^2=3\pi r^2$.
Hence o' ≏ 3×o. (W.,Art.,463)

157. To construct a circle equivalent to three-fourths of a given circle.

Let o be the given circle.

To construct a circle equivalent to three-fourths of o.

Let r be the radius of the given circle.



Take r' of (Art.,156) and construct $r''=1/2r'=1/2r\sqrt{3}$. (Art.,14)
Then a circle, o", described with r" as a radius will be the required circle.

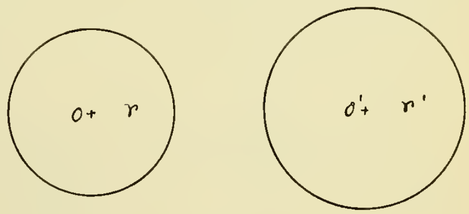
Proof: Squaring $r''=1/2r\sqrt{3}$ and multiplying by π we get
$\pi r''^2=3\pi r^2/4$. Hence o" ≏ 3/4o. (W.,Art.,463)

158. To construct a circle whose ratio to a given circle shall be equal to the given ratio m:n.

Let o be the given circle and m:n the given ratio.

To construct a circle o' such that m:n=o:o'.

Let r be the radius of the given circle and construct r^2. (Art.,8,Cor.)



Construct r'^2 such that $m:n=r^2:r'^2$. (Art.,6)
From r'^2 construct r'. (Art.,10)
Then construct a circle with radius r'. Then this is the required circle.

Proof: $m:n=r^2:r'^2$. Const.
Hence m:n=o:o'. (W.,Art.,464)

159. To divide a given circle by a concentric circumference into two equivalent parts.

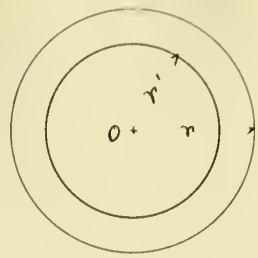
Let r be the radius of the given circle.

To construct a circumference with o as a centre which shall divide the given circle into two equivalent parts.

Construct $\sqrt{2}$. (Art.,10)

Then construct r' such that $r'=r/\sqrt{2}$. (Art.,9)

Then a circle constructed with r' as a radius will be the required circle.

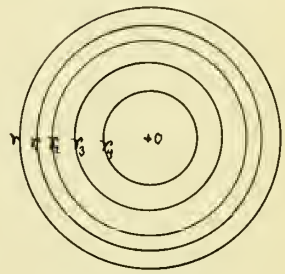


Proof: Squaring $r'=r/\sqrt{2}$ and multiplying by π we get $\pi r'^2=\pi r^2/2$. Hence the circle with radius $r' \simeq$ 1/2 the circle with radius r. (W.,Art.,463) Therefore the circle with radius r is divided into two equivalent parts by the circumference of r'.

160. To divide a given circle by concentric circumferences into five equivalent parts.

Let o be the centre of the given circle, and r its radius.

To divide o by concentric circumferences into five equivalent parts.



Construct $\sqrt{5}$. (Art.,10)

Also construct $r/\sqrt{5}$ (Art.,9)

Then $\sqrt{2}\times r/\sqrt{5}$, $\sqrt{3}\times r/\sqrt{5}$, $\sqrt{4}\times r/\sqrt{5}$. (Art.,8)

Construct concentric circumferences with radii $r_4=r/\sqrt{5}$, $r_3=\sqrt{2}\times r/\sqrt{5}$, $r_2=\sqrt{3}\times r/\sqrt{5}$, $r_1=2\times r/\sqrt{5}$. Then these are the required circumferences.

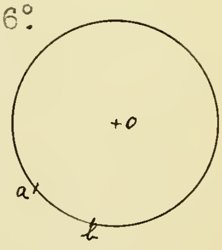
Proof: The areas of the respective circles are πr^2, $\pi r^2/5$, $2\pi r^2/5$, $3\pi r^2/5$, $4\pi r^2/5$. (W.,Art.,463)

Hence these are the required circumferences.

161. To construct an angle of 36°.

Let o be any circle and construct ab as the side of a regular inscribed decagon. (Art.,143)

Then aob is the required angle.



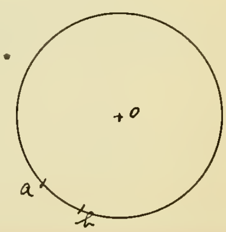
Proof: $\angle aob$ =1/10 of 360°=36°. (W.,Arts.,243;237)

*Cor. Bisecting $\angle aob$ we get an angle of 18° and bisecting this again we get an angle of 9°. (Art.,15)

162. To construct an angle of 24°.

Let o be any circle and construct ab as the side of a regular inscribed pentadecagon. (Art.,144)

Then aob is the required angle.



Proof: $\angle aob = 1/15$ of $360° = 24°$. (W., Arts., 243, 237)

*Cor. Bisecting $\angle aob$ we get an angle of 12° and bisecting this again we get an angle of 6°. (Art. 15)

163. To construct with a side of a given length;

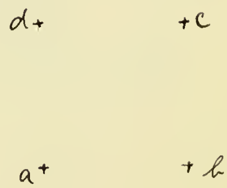
I. An equilateral triangle.

Let ab be the given side.

To construct an equilateral triangle having ab as a side.

With a and b as centres and ab as a radius describe arcs intersecting at c. Then abc is the required triangle.

II. A square.



Let ab be the given side.

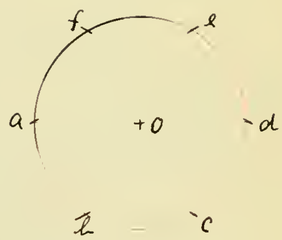
To construct a square having a side ab.

At a and b construct bc and ad perpendicular to ab. (Art., 12, I)

Lay off ad=bc=ab. (Art., 4, Cor.)

Then abcd is the required square.

III. A regular hexagon.



Let oa be the given side.

To construct a regular hexagon with oa as a side.

Draw a circle, o, with oa as a radius and within it inscribe a regular hexagon. (Art., 142)

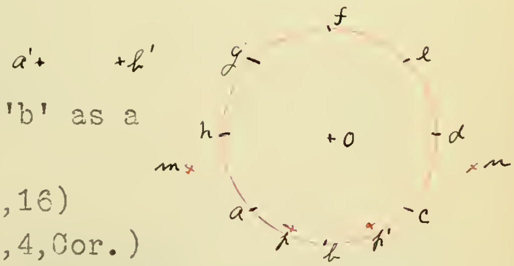
Then abcdef is the required hexagon.

Proof: ab=bc=cd=de=ef=fa. Const.

Also ab=oa. Const.

Therefore abcdef is the required hexagon.

IV. A regular octagon.



Let a'b' be the given side.

To construct a regular octagon having a'b' as a side.

Construct $\angle mbn = 135°$. (Art., 16)

Lay off ab=bc=a'b'. (Art., 4, Cor.)

Construct perpendiculars, po and p'o, at the middle points of ab and bc and find their intersection o. (Arts., 12, 7)

Construct a circle with o as a centre and oa as a radius and lay off the chords ab=bc=cd=de.

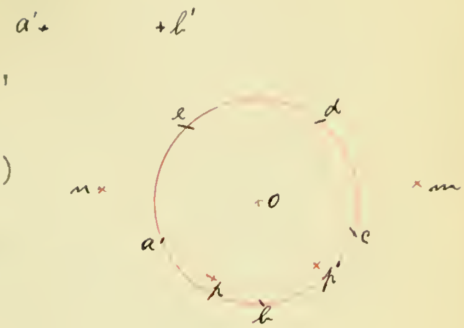
Then abcde... is the required octagon.

Proof: $\angle abc = 135°$. Const.

Hence $\angle aob=45^\circ$. (W., Art., 438)

Therefore acbde...is a regular octagon.

V. A r e g u l a r p e n t a g o n .



Let a'b' be the given side.

To construct a regular pentagon having a'b' as a side.

Construct $\angle nbm=108^\circ$. (Art., 143, Cor, I)

Lay off ab=bc=a'b'. (Art., 4, Cor.)

Construct perpendiculars, op and op', at the middle points of ab and bc and find their intersection o. (Arts., 12, 7)

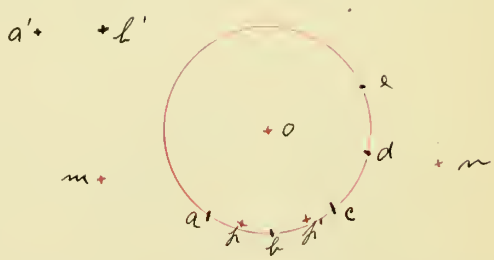
Construct a circle with o as a centre and ao as a radius and lay off chords ab=bc=cd=de. Then abcde is the required pentagon.

Proof: $\angle abc=108^\circ$. Const.

Hence $\angle aob=72^\circ$. (W., Art., 438)

Therefore abcde is a regular pentagon.

VI. A r e g u l a r d e c a g o n .



Let a'b' be the given side.

To construct a regular decagon having a'b' as a side .

Construct $\angle mbn=144^\circ$. (Art., 143)

Lay off ab=bc=a'b'. (Art., 4, Cor.)

Construct perpendiculars, op and op', at the middle points of ab and bc and find their intersection o. (Arts., 12, 7)

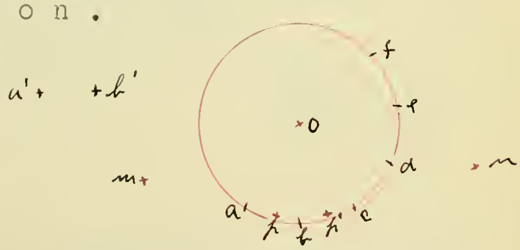
Construct a circle with o as a centre and oa as a radius and lay off chords ab=bc=cd=etc. Then abcd...is the required decagon.

Proof: $\angle abc=144^\circ$. Const.

Hence $\angle aob=36^\circ$. (W., Art., 438)

Therefore abcd..is the required decagon.

VII. A r e g u l a r d o d e c a g o n .



Let a'b' be the given side.

To construct a regular dodecagon having a'b' as a side.

Construct $\angle mbn=150^\circ$. (Art., 17, Cor)

Lay off ab=bc=a'b'. (Art., 4, Cor.)

Construct perpendiculars, po and p'o, at the middle points of ab and bc and find their intersection o.. (Art., 12, 7)

Construct a circle with o as a centre and oa as a radius and lay off the chords ab=bc=cd=etc. Then abcd...is the required dodecagon.

Proof: $\angle abc=150^\circ$. Const.

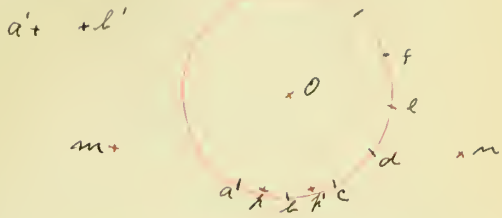
Hence $\angle aob=30^\circ$. (W., Art., 438)

Therefore abcd...is the required dodecagon.

VIII. A regular pentedecagon.

Let a'b' be the given side.

To construct a regular pentedecagon having a'b' as a side.



Construct $\angle$mbn=156°. (Art.,144)

Lay off chords ab=bc=a'b'. (Art.,4,Cor.)

Construct perpendiculars,po and p'o,at the middle points of ab and bc and find their intersection o. (Arts.,12,7)

Construct a circle with o as a centre and oa as a radius and lay off chords ab=bc=etc. Then abcd... is the required pentedecagon.

Proof: $\angle$abc=156°. Const.

Hence $\angle$aob=24°. (W. Art.,438)

Therefore abcd...is the required pentedecagon.

164. To divide a given trapezoid into two equivalent parts by a line parallel to the bases.

Let b"c"cb be the given trapezoid.

To divide b"c"cb into two equivalent parts by a line parallel to b"c".

Suppose b'c' is the required line.

Find the intersection,a,of b"b and c"c. (Art.,7)

Construct ad" perpendicular to b"c" and find its intersections,d and d',with bc and b'c'. (Arts.,12,7)

Tri,abc≎(ad×bc)÷2 and trapezoidb"c"cb≎((b"c"+bc)÷2)dd". (W.,Arts403,407)

Then tri.ab'c'≎((ad+bc)÷2)+1/2((b"c"+bc)÷2)dd"≎(2ad×bc+(b"c"+bc)dd")÷4. Const.

Triangles abc and ab'c' are similar. (W.,Art.,354)

Therefore((ad×bc)÷2):(2ad×bc+(b"c"+bc)dd")÷4)=$\overline{ad}^2:\overline{ad'}^2$. (W.,Art.,412)

Solving this equation we get ad'=$\sqrt{\overline{ad}^2+(ad(bc+b"c")dd")\div 2bc}$.

Therefore,construct ad'=$\sqrt{\overline{ad}^2+(ad(bc+b"c")dd")\div 2bc}$. (Arts.,5,8,9,10)

Through d',construct b'c' parallel to b"c". (Art.,20)

Then b'c' is the required line.

165. To divide a given trapezoid into two equivalent parts by a line through a given point in one of the bases.

Let abcd be the given trapezoid and p the given point in one of the bases.

To draw a line through p dividing abcd into two equivalent parts.

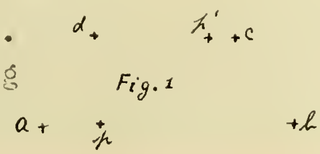


Fig. 1

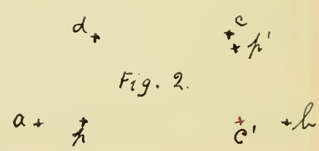


Fig. 2

Suppose pp' the line (Fig.I) For brevity let dc=m, p'c=x, ap=n, pb=r dp=l. Since the two trapezoids app'd and pbcp' are equivalent and have the same altitude, we must have $x+r=m-x+n$. (W., Art., 407)
Hence $x=(m+n-r)\div 2$.
Therefore construct $cp'=(dc+ap-pb)\div 2$ (Art., 5, 9)
Then pp' will be the required line.

*Cor. If x is negative in this construction one of the equivalent figures will be a triangle. Suppose pp' the required line. (Fig.2) Let p'c'=x and the other values be the same as above. Since the triangle pp'b is equivalent to half the trapezoid abcd we have

$$((m+n+r)\div 4)l=1/2rx \quad \text{or} \quad x=((m+n+r)\div 2r)l$$

Therefore construct $c'p'=((cd+ap+pb)\div 2pb)dp$ (Arts., 5, 8, 9)
Then pp' is the required line.

166 To construct a regular pentagon, given one of the diagonals.

Let ac be the given diagonal.

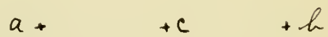
To construct a regular pentagon with ac as a diagonal.

Construct ∠oac=∠oca=18°. (Art., 161, Cor.)
With o as a centre and oa as a radius, describe a circle.
Inscribe a regular pentagon in this circle. (Art., 143, Cor.)
Then this is the required pentagon.

Proof: ∠aoc=180°-36°=144°. (W., Art., 129)
Since ac is a diagonal, the arc ac must be subtended by two equal chords which are the sides of the required pentagon. Bisect ∠aoc. (Art., 15)
Then ∠aob=72°=1/5 of ~~the~~ right angles. Hence abcd is the required pentagon.

167. To divide a given straight line into two segments such that their product shall be the maximum.

Let ab be the given line.



To divide ab into two segments such that their product shall be the maximum.

Bisect ab at c. (Art., 14)

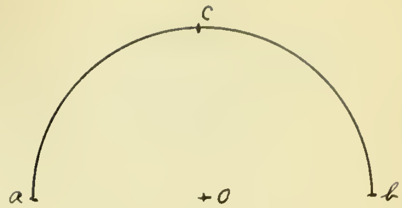
Then ac and bc are the required segments. (W., Art., 489)

168. To find a point in a semicircumference such that the sum of its distances from the extremities of the diameter shall be

the maximum.

Let acb be the given semicircumference.

To find a point in acb such that the sum of its distances from a and b shall be the maximum.

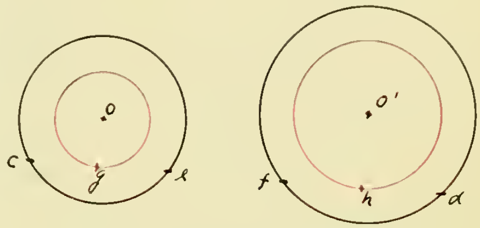


Bisect the arc acb at c. (Art.,1)

Then c is the required point. (W.,Art.,485)

169. To draw a common secant to two given circles exterior to each other such that the intercepted chords shall have the given lengths a, b.

Let the outer circles whose centres are o and o' be the given circles.



To draw a common secant to o and o' such that the intercepted chords shall have the given lengths a, b.

Construct $og=\sqrt{oe^2-a^2/4}$ (Arts.,5,8,10)

Then construct a circle with o as a centre and og as a radius .

Construct $o'h=\sqrt{of^2-b^2/4}$, (Arts.,5,8,10)

and then construct a circle with o' as a centre and o'h as a radius.

Construct a common tangent to the two inner circles whose centres are o and o'. (Art.,84)

Then this is the required secant.

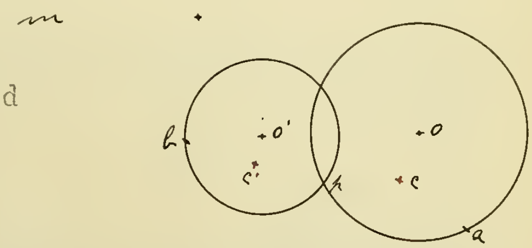
Proof: cg=ge and fh=hd. (W.,Art.,245)

ce=a and fd=b. (W.,Art.,249)

Therefore cd is the required secant.

170. To draw through one of the points of intersection of two interesecting circles a common secant which shall be of given length.

Let o and o' be the given circles, p their point of intersection and m the length of the required secant.



To draw a common secant through p having the given length m.

Construct the right triangle oo'c having hypotenuse=oo' and leg o'c=m/2

Construct ab parallel to o'c and through the point .. (Art.,20)

Then ab is the required secant.

Proof: oc and o'c' are perpendicular to ab. Const.

Hence oc bisects ap and o'c' bisects pb. (W., Art., 245)

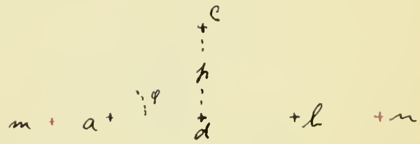
But $o'c = m/2$. Const.

Therefore ab is the required secant.

171. To construct an isosceles triangle given the altitude and one of the equal base angles.

Let φ be the given angle and p the given altitude.

To construct an isosceles triangle with altitude p and φ as one of the equal angles.



On any line mn construct cd perpendicular to mn. (Art., 12)

Lay it off equal to p. (Art., 4, Cor.)

Construct $\angle dca = \angle dcb = 180° - (\varphi + 90°) = 90° - \varphi$

and find the points, a and b, where the lines ca and cb intersect mn. (Arts., 19, 7)

Then abc is the required triangle.

Proof: $cd = p$ and $\angle cad = \angle cbd = \varphi$. Const.

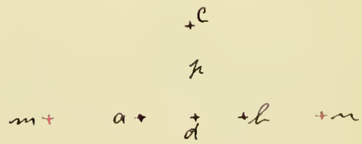
Therefore abc is an isosceles triangle. (W., Art., 120)

Therefore abc is the required triangle.

172. To construct an equilateral triangle, given the altitude.

Let p be the given altitude.

To construct an equilateral triangle having p as the altitude.



On any line mn, construct cd perpendicular to mn. (Art., 12)

Lay it off equal to p. (Art., 4, Cor.)

Construct $\angle dca = \angle dcb = 30°$

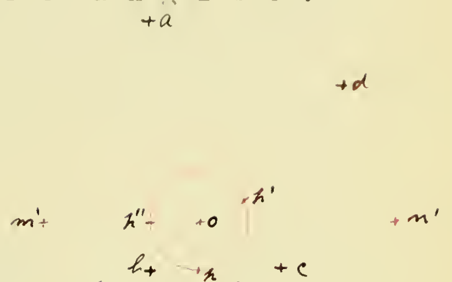
and find the points, a and b, where the lines ca and cb intersect mn. (Arts., 18, 7)

Then abc is the required triangle.

Proof:	$cd = p$.	Const.
	tri. adc = tri. bdc.	(W., Art., 139)
Hence	$\angle cad = \angle cbd$.	(W., Art., 128)
But	$\angle acb = 60°$.	Const.
Hence	$\angle acb = \angle cab = \angle cba$,	

Then triangle abc is equilateral. (W., Art., 148)
Therefore abc is the required triangle.

173. To construct a right triangle, given the radius of the inscribed circle and the difference of the acute angles.



Let m be the radius of the inscribed circle and φ the difference of the acute angles.

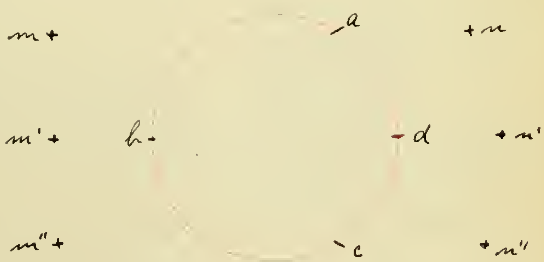
To construct a right triangle whose inscribed circle shall have a radius m and the difference of whose acute angles is φ.

Construct ab perpendicular to bc. (Art., 12)
Construct the bisector, bd, of the angle abc. (Art., 15)
At a distance m from bc, construct m'n' parallel to bc. (Art., 20)
Find the intersection o of m'n' and bd. (Art., 7)
With o as a centre and radius m, describe a circle. This will be tangent to ab and bc. (W., Art., 162)

Construct $\angle pop' = (270° - \varphi) \div 2$. (Art., 19)
Construct a tangent to the circle at p'. (Art., 36, I)
Find its intersections, c and a, with bc and ab respectively. (Art., 7)
Then abc is the required triangle.

Proof: $\angle abc = 90°$ and $pop' = (270° - \varphi) \div 2$. Const.
Hence $\angle c = 180° - ((270° - \varphi) \div 2) = (90° + \varphi) \div 2$. (W., Art., 205)
$\angle p'op'' = 270° - ((270° - \varphi) \div 2) = (270° + \varphi) \div 2$. (W., Art., 88)
Hence $\angle a = 180° - ((270, + \varphi) \div 2) = (90° - \varphi) \div 2$. (W., Art., 205)
Then $\angle c - \angle a = ((90° + \varphi) \div 2) - ((90° - \varphi) \div 2) = \varphi$.
Therefore abc is the required triangle.

174. To construct an equilateral triangle so that its vertices shall lie in three given parallel lines.



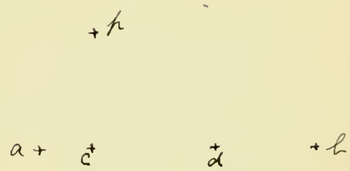
Let mn, m'n', m"n" be the three given lines.

To construct an equilateral triangle having its vertices on the given lines.

Let abc be the required triangle.
Circumscribe a circle about the triangle abc and find its intersection, d, with m'n'. (Arts., 34; 4, I)
Then $\angle adb = \angle bdc = 60°$. (W., Art., 189)
Hence, to construct abc take any point d in m'n' and construct $\angle bda = \angle bdc = 60°$

(17,Cor.)
Find the intersections,d and c,of mn and ad,m"n" and dc. (Art.,7)
Then construct an equilateral triangle with ac as one side. (Art.,17)
Then abc is the required triangle.

175. To draw a line from a given point to a given straight line which shall be to the perpendicular from the given point as m:n.



Let p be the given point,ab the given line and m:n the given ratio.

To construct a line pd such that m:n=pc:pd.

Construct pc perpendicular to ab. (Art.,12)
Find the intersection,c,of ab and pc. (Art.,7)
Construct pd such that m:n=pc:pd. (Art.,6)
With p as centre and pd as a radius strike an arc cutting ab at d.
(Art.,4,II)
Then pd is the required line.

176. To find a point within a given triangle such that the perpendiculars from the point to the three sides shall be as the numbers m,n,p.

Let abc be the given triangle and m,n,p, the given numbers

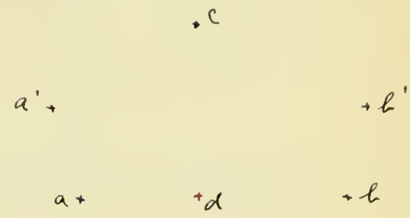
To find a point within abc such that its distances from the sides of the triangle shall be as m,n and p.

Construct m'n' parallel to cb and at a distance n from it and find their intersection p'. (Arts.,20,7)
Construct p'o perpendicular to ab
and lay it off equal to the given length p. (Arts.,12;4,Cor.)
Through o,construct a'b' parallel to ab. (Art.,20)
Construct r and s such that cb':cb=m:r and ca':ca=n:s. (Art.,6)
Construct lines parallel to cb and ca and at distances r and s from them and find their intersection p". (Arts.,20,7)
Let the distance from p" to ab be t. Then a'b':ab=p:t. (W.,Art.,352)
Hence p" is the required point.

Proof: Tri.abc and a'b'c are similar. (W.,Art.,354)
Hence cb':cb=ca':ca=a'b':ab. (W.,Art.,351)
Therefore m:r=n:s=p:t or m:n:p=r:s:t. (W.,Art.,330)

Therefore p" is the required point.

177. To draw a straight line equidistant from three given points.



Let a,b and c be the three given points.

To draw a line equidistant from a,b and c.

Construct cd perpendicular to ab,(Art.,12,II) and find its intersection ,d,with ab. (Art.,7) Construct a'b' perpendicular to cd at its middle point. (Art.,12)

Then a'b' is the required line.

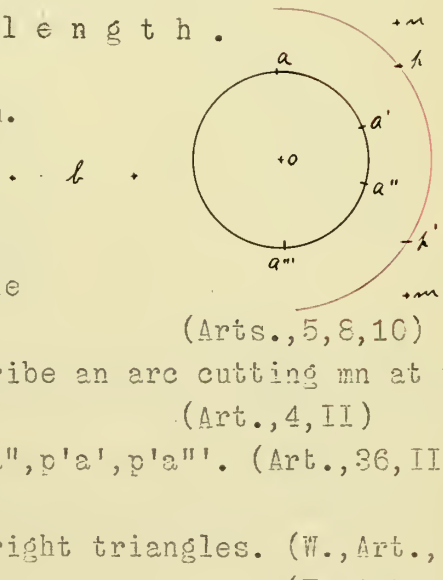
Proof: a'b' is parallel to ab. (W.,Art.,104)

Therefore the distance of any point inab from a'b' is the same. (W.,Art.,181)

Every point in a'b' is equidistant from c and d. (W.,Art.,160)

Therefore c,a and b are equidistant from a'b'. Hence a'b' is the required line.

178. To draw a tangent to a given circle such that the segment intercepted between the point of contact and a given straight line shall have a given length.



Let o be the given circle,mn the given straight line and b the given length.

To construct a tangent to o such that the part included between mn and the point of contact shall have the length b.

Let r be the radius of the given circle and construct $R=\sqrt{r^2+b^2}$. (Arts.,5,8,10)

With o as a centre and R as a radius,describe an arc cutting mn at p andp' (Art.,4,II)

From p and p' construct the tangents pa,pa",p'a',p'a"'. (Art.,36,II)

Then these are the required tangents.

Proof: Triangles oap,oa'p' etc.,are right triangles. (W.,Art.,254)

Then $R^2-r^2=\overline{ap}^2$. (W.,Art.,372)

Hence $r^2+b^2-r^2=\overline{ap}^2$ or $b^2=\overline{ap}^2$ or b=ap. Similarly for the other tangents.

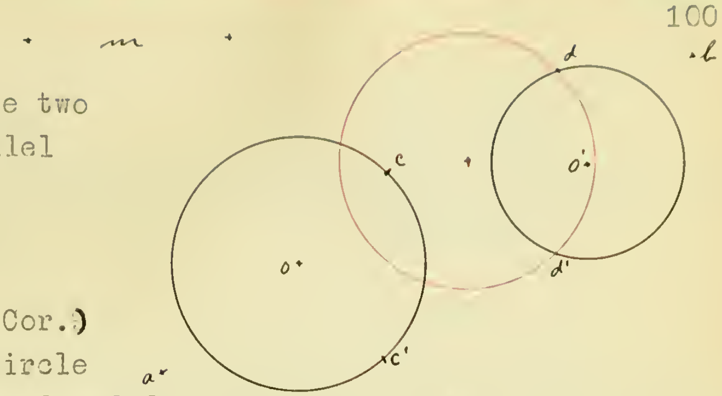
Therefore these are the required tangents.

178. To inscribe a straight line of a given length between two given circumferences and parallel to a given straight line.

Let o and o' be the given circles,m the given length and ab the given

line.

To inscribe a line between the two circumferences of o and o', parallel to ab and equal to m.



Construct oo" parallel to ab. (Art.,20)

Lay it off equal to m. (Art.,4,Cor.)

With o" as a centre describe a circle having the same radius as o. Let d and d' be the points where this circumference cuts the circumference of o'.

Construct dc and d'c' parallel to ab. (Art.,20)

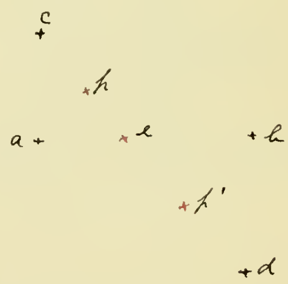
Then dc and d'c' are the required lines.

Proof: Since the distance between the centres o,o" of the two equal circles is m, then the parallel distance between any two points on the circumferences is m. Then dc=d'c'=m.

Therefore dc and d'c' are the required lines.

179. To draw through a given point a straight line so that its distances from two other given points shall be in a given ratio.

Let a,b and c be the given points and m:n the given ratio.



To draw a line through c such that its distances from a and b shall be in the ratio m:n.

Divide ab at e in the ratio m:n. (Art.,85)

Extend ce to d. (Art.,4,Corl)

Then cd is the required line.

Proof: Construct ap and bp' perpendicular to cd. (Art.,12,11)

Then triangles ape and bp'e are similar. (W.,Art.,356)

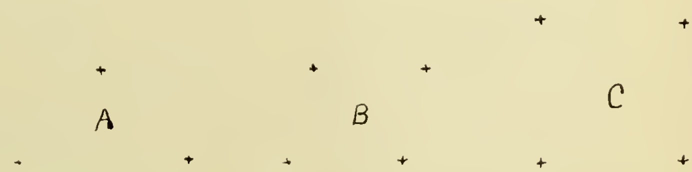
But ae:be=m:n. Const.

Hence ap:bp'=m:n. (W.,Art.,356)

Therefore cd is the required line.

180. To construct a square equivalent to the sum of a given triangle and a given parallelogram.

Let A be the given triangle and B the given parallelogram.



To construct a square

equivalent to A+B.

Construct a square equivalent to A. (Art.,119,Cor.)
Construct a square equivalent to B. (Art.,119)
Construct a square,C,equivalent to the sum of the two squares already constructed. (Art.,115)
Then C is the required square.

181. *To construct a rectangle having the difference of its base and altitude equal to a given line, and its area equivalent to the sum of a given triangle and a given pentagon.*

Let A be the given triangle, B the given pentagon and m the difference between the base and altitude of the required rectangle.

To construct a rectangle equivalent to A+B, the difference of whose base and altitude is m.

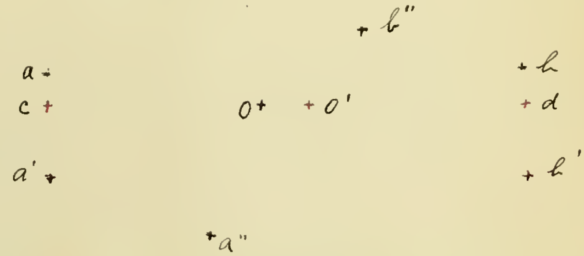
Construct a square A' equivalent to A. (Art.,119,CorI)
Construct a triangle B' equivalent to B. (Art.,118)
Then construct a square B" equivalent to B'. (Art.,119,Cor.I)
Construct a square D equivalent to (A'+B"). (Art.,115)
Finally construct a rectangle C equivalent to the square D and having the difference of its base and altitude equal to m. (Art.,121)
Then C is the required rectangle.

182. *To construct a pentagon similar to a given pentagon and equivalent to a given trapezoid.*

This is a special case of Art.,122. Both of the given polygons must be reduced to squares and then the required polygon can be constructed.

183. *To find a point whose distances from three given straight lines shall be as the numbers m,n,p.*

Let ab,a'b' and a"b" be the three given straight lines and m,n,p the given numbers.



To find a point whose distances from ab,a'b' and a"b" shall be as the

numbers m,n,p.

Locate c such that ac:a'c=m:n. (Art.,85)
Construct cd parallel to ab. (Art.,20)
Find the intersection,o',of cd and a"b". (Art.,7)
Lay off o'o such that n:p=ca':o',. (Art.,6)
Then o is the required point.

Proof: ac:a'c=m:n and a'c:o'o=n:p. Const.
Multiplying one equation by the other we get ac:o'o=m:p-
Hence ac:a'c:o'o=m:n:p.
Therefore o is the required point.

*Cor.I. If the three straight lines form a triangle this exercise reduces to Art.,176.

*Cor.II. If the three lines are parallel the solution is impossible except for particular values of m,n and p.

184. Given an angle and two points P and P' between the sides of the angle. To find the shortest path from P to P' that shall touch both sides of the angle.

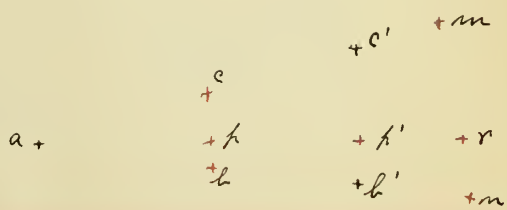
Let abc be the given angle and P and P' the given points.

To find the shortest path from P to P' that shall touch both ab and bc.

Construct Po perpendicular to bc and P'o' perpendicular to ab. (Art.,12,II)
Extend Po to p and P'o' to p' making P'o'=o'p' and Po=op. (Art.,4,Cor.)
Find the intersections,m and m',of pp' with bc and ab respectively. (Art.,7)
Then Pmm'P' is the required path.

Proof: Pm=pm and P'm'=p'm'. (W.,Art.,160)
Hence Pmm'P'=pp'. If we should take any other points,n and n', then Pnn'P'=pnn'p'. But pnn'p' is greater that pmm'p' or pp'. (W.,Art.,49)
Hence Pnn'P'is greater than pp' or Pnn'P' is greater than Pmm'P'.
Therefore Pmm'P' is the required path.

185. To construct a triangle, given its angles and its area.



Let A be the given area and bac,abc,acb the given angles.

To construct a triangle with area A and

angles bac, abc and acb.

Construct any triangle, abc, having the given angles. (Art., 19)
Construct ap perpendicular to bc. (Art., 12, II)
Extend ac to m, ab to n and ap to r. (Art., 4, Cor.)
Construct $ap' = \sqrt{(2A \times ap) \div bc}$ (Arts., 8, 9, 10)
Through p' construct b'c' parallel to bc
and find its intersections, c' and b', with am and an. (Arts., 20, 7)
Then ab'c' is the required triangle.

Proof: Triangles abc and ab'c' are similar. (W., Art., 354)
Hence $\text{tri.}abc : \text{tri.}ab'c' = \overline{ap}^2 : \overline{ap'}^2$. (W., Art., 412)
But $\text{tri.}abc = 1/2\ ap \times bc$, (W., Art., 403)
and $\overline{ap'}^2 = (2A \times ap) \div bc$. Const.
Then $\text{tri.}ab'c' \times \overline{ap}^2 = 1/2\ ap \times bc \times ((2A \times ap) \div bc)$ or $\text{tri.}ab'c' \approx A$.
Therefore ab'c' is the required triangle.

186. To transform a given triangle into a triangle similar to another given triangle.

This is a special case of Art., 122. The triangles are reduced by Art., 119, Cor. I.

187. Given three points A, B, C. To find a fourth point P such that the areas of the triangles APB, APC, BPC shall be equal.

This is another statement of Art., 138 with the exception of the case where A, B and C lie in the same straight line, in which case the solution is impossible.

188. To construct a triangle, given its base, the ratio of the other sides and angle included by them.

Let m be the given base, n:p the ratio of the other sides and bac the included angle.

To construct a triangle with base m, the other two sides having the ratio n:p and an included angle bac.

Lay off ab'=n and ac'=p. (Art., 4, Cor.)
Construct ac such that b'c':ac'=m:ac. (Art., 6)
From c, construct cb parallel to c'b'.
and find its intersection, b, with ab. (Arts., 20, 7)

Then abc is the required triangle.

Proof: bac is the given angle. Const.

Triangles abc and ab'c' are similar. (W., Art., 357)

Then ab':ac'=ab:ac=n:p and b'c':ac'=bc:ac. (W., Art., 351)

But b'c':ac'=m:ac. Const.

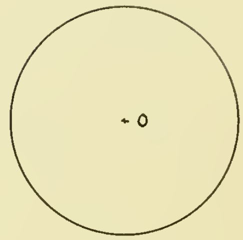
Hence bc=m.

Therefore abc is the required triangle.

189. To divide a given circle into equivalent parts by concentric circumferences.

Let o be the given circle and r its radius.

To divide the circle o into n equal parts by concentric circumferences.



Construct $r'=r\sqrt{1/n}$, $r''=r\sqrt{2/n}$, $r'''=r\sqrt{3/n}$ etc. (Arts., 8, 9, 10)

Construct circles with o as a centre and r', r'', r''' as radii.

Then these are the required circumferences.

Proof: The areas of these separate circles are $\pi r'^2=\pi r^2/n$, $\pi r''^2=2\pi r^2/n$ $\pi r'''^2=3\pi r^2/n$ etc. (W., Art., 468)

The difference between the areas of any two of these circles is $\pi r^2/n$.

Therefore the original circle is divided into n equal parts.

190 In a given equilateral triangle to inscribe three equal circles tangent to each other, each circle tangent to two sides of the triangle.

Let abc be the given triangle.

To inscribe three equal circles in abc tangent to each other and each tangent to two sides of abc.

Bisect angles a, b and c by aa', bb', cc'. (Art., 15)

These intersect in a point p. (W., Ex, 24)

Let c'f bisect the angle bc'c and find the intersection, o, of c'f and bb'. (Art., 7)

Construct oe perpendicular to ab. (Art., 12, II)

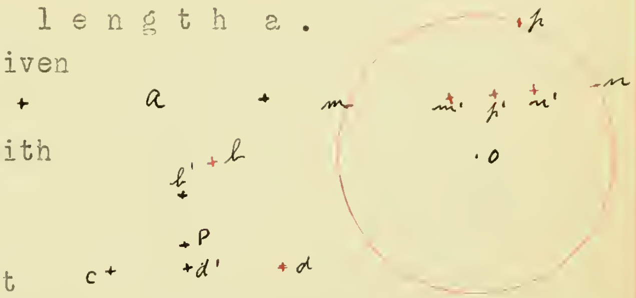
With o as a centre and oe as a radius, describe a circle.

Make po'=po''=po. (Art., 4, Cor.)

With o' and o'' as centres and the same radius as before, describe the other two circles. Then o, o' and o'' are the required circles.

Proof: o is equidistant from, bc', c'p, pa' and a'b. (W., Art., 162)
Hence the circles are tangent to the sides of the triangle.
In the right triangle opd and o'pd' (W., Art., 254
op=o'p and o'd'=od. Const.
Then triangle opd=triangle o'pd'. (W., Art., 151)
Hence pd=pd' and the circles are tangent at d. Similarly the other circles are tangent to each other. Therefore o, c' and o" are the required circles..

191. Given an angle and a point P between the sides of the angle. To draw through P a straight line that shall form with the sides of the angle a triangle with perimeter equal to a given length a.



Let bcd be the given angle, P the given point and a the given perimeter.

To draw a line through P forming with the sides of the given angle a triangle whose perimeter is a.

Take a line mn equal to a and on it construct the segment of a circle in which an angle, $90°+1/2\angle bcd$, may be inscribed. (Art., 38)
With any point, p, in the arc mpn as a centre and a radius Pc, describe an arc cutting mn at p'. (Art., 4, I)
Erect perpendiculars at the middle points of mp and np and find their intersections m' and n' with mn. (Arts., 12, 7)
Lay off cb'=pm'. (Art., 4, Cor.)
Find the intersection d' of cd and b'P. (Art., 7)
Then ab'd' is the required triangle.

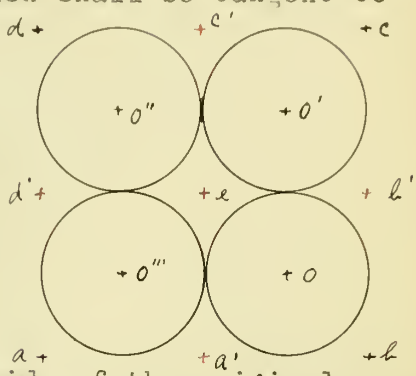
Proof: mm'=pm' and nn'=pn'. (W., Art., 160)
Hence the perimeter of triangle pm'n'=a.
$\angle pmm'=\angle mpm'$ and $\angle n'pn=\angle pnn'$. (W., Art., 145)
$\angle mpn=90°+1/2\ \angle bcd$. Const.
$\angle pmm'+pnn'+90°+1/2\ \angle bcd=180°$ or $\angle pmm'+pnn'+1/2\angle bcd=90°$. (W., Art., 129)
$\angle mpn=90°+1/2\ \angle bcd=\angle pmm'+\angle pnn'+\angle m'pn'$
or $\angle pmm'+\angle pnn'=90°+1/2\angle bcd-\angle m'pn'$.
Substituting this in the equation above we get
$90°+1/2\angle bcd-\angle m'pn'+1/2\angle bcd=90°$ or $\angle m'pn'=\angle bcd$.
Hence triangle m'pn'=triangle b'cd'. (W., Art., 143)
Therefore the perimeter of triangle b'cd'=a and it is the required triangle.

192. In a given square to inscribe four

equal circles, so that each circle shall be tangent to two of the others and also tangent to two sides of the square.

Let abcd be the given square.

To inscribe four circles in abcd each of which shall be tangent to two others and tangent to two sides of the square.



Divide the given square into four equal squares by erecting perpendiculars at the middle points of the sides. (Art., 12, I)

Find the intersections of the diagonals of the smaller squares. (Art., 7)

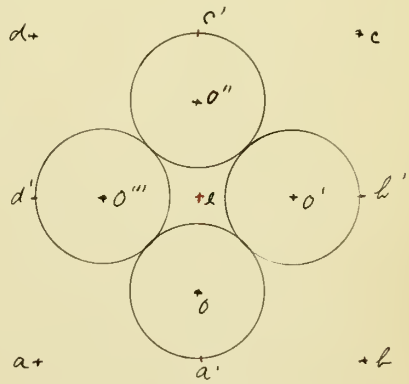
With these points as centres and one fourth the side of the original square as radius, describe circles. Then these are the required circles.

Proof: Since the diagonals of a square bisect its angles their point of intersection is equidistant from all four sides. Hence each circle is tangent to the sides of the smaller squares at their middle points. Hence each circle is tangent to two others. Therefore o, o', o", o'" are the required circles.

193. In a given square to inscribe four equal circles so that each circle shall be tangent to two of the others and also tangent to one side of the square.

Let abcd be the given square.

To inscribe four circles in abcd such that each shall be tangent to two of the others and to one side of the square.



Find the intersection e of ac and bd. (Art., 7)

Inscribe circles in the triangles ade, dec, ceb and aeb. (Art., 35)

Then these are the required circles.

Proof: Each is tangent to one side of the square. Const. Since all the triangles are equal, the distance from any vertex of the square to the point of tangency of any of the circles is the same. Hence each circle is tangent to two of the others. Therefore o, o', o" and o'" are the required circles.

I n d e x .